Introduction to Algorithmic Trading Strategies
Lecture 8

Risk Management

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Outline

- Value at Risk (VaR)
- Extreme Value Theory (EVT)

References

- AJ McNeil. Extreme Value Theory for Risk Managers. 1999.
- Blake LeBaron, Ritirupa Samanta. Extreme Value Theory and Fat Tails in Equity Markets. November 2005.

Risks

- Financial theories say:
  - the most important single source of profit is risk.
  - profit $\propto$ risk.
- *I personally do not agree.*

What Are Some Risks? (1)

- Bonds:
  - duration (sensitivity to interest rate)
  - convexity
  - term structure models
- Credit:
  - rating
  - default models

What Are Some Risks? (2)

- Stocks
  - volatility
  - correlations
  - beta
- Derivatives
  - delta
  - gamma
  - vega

What Are Some Risks? (3)

- FX
  - volatility
  - target zones
  - spreads
  - term structure models of related currencies

Other Risks?

- Too many to enumerate…
  - natural disasters, e.g., earthquake
  - war
  - politics
  - operational risk
  - regulatory risk
  - wide spread rumors
  - alien attack!!!
- Practically infinitely many of them…

VaR Definition

- Given a loss distribution, F, quintile $1 > \mathrm{q} \geq 0.95$,
- $\mathrm{VaR}_q = F^{-1}(q)$

Expected Shortfall

- Suppose we hit a big loss, what is its expected size?
- $\mathrm{ES}_q = E\left[X | X > \mathrm{VaR}_q\right]$

VaR in Layman Term

- VaR is the maximum loss which can occur with certain confidence over a holding period (of n days).
- Suppose a daily VaR is stated as $1,000,000 to a 95% level of confidence.
- There is only a 5% chance that the loss the next day will *exceed* $1,000,000.

Why VaR?

- Is it a true way to measure risk?
  - NO!
- Is it a universal measure accounting for most risks?
  - NO!
- Is it a good measure?
  - NO!
- Only because the industry and regulators have adopted it.
  - It is a widely accepted standard.

VaR Computations

- Historical Simulation
- Variance-CoVariance
- Monte Carlo simulation

Historical Simulations

- Take a historical *returns* time series as the returns distribution.
- Compute the loss distribution from the historical returns distribution.

Historical Simulations Advantages

- Simplest
- Non-parametric, no assumption of distributions, no possibility of estimation error

Historical Simulations Dis-Advantages

- As all historical returns carry equal weights, it runs the risk of over-/under- estimate the recent trends.
- Sample period may not be representative of the risks.
- History may not repeat itself.
- Cannot accommodate for new risks.
- Cannot incorporate subjective information.

Variance-CoVariance

- Assume all returns distributions are Normal.
- Estimate asset variances and covariances from historical data.
- Compute portfolio variance.
  - ${\sigma_P}^2 = \sum_{i,j} \rho_{ij}\omega_i\omega_j\sigma_i\sigma_j$

Variance-CoVariance Example

- 95% confidence level (1.645 stdev from mean)
- Nominal = $10 million
- Price = $100
- Average return = 7.35%
- Standard deviation = 1.99%
- The VaR at 95% confidence level = 1.645 x 0.0199 = 0.032736
- The VaR of the portfolio = 0.032736 x 10 million = $327,360.

Variance-CoVariance Advantages

- Widely accepted approach in banks and regulations.
- Simple to apply; straightforward to explain.
- Datasets immediately available
  - very easy to estimate from historical data
  - free data from RiskMetrics
  - http://www.jpmorgan.com
- Can do scenario tests by twisting the parameters.
  - sensitivity analysis of parameters
  - give more weightings to more recent data

Variance-CoVariance Disadvantages

- Assumption of Normal distribution for returns, which is known to be not true.
- Does not take into account of fat tails.
- Does not work with non-linear assets in portfolio, e.g., options.

Monte Carlo Simulation

- You create your own returns distributions.
  - historical data
  - implied data
  - economic scenarios
- Simulate the joint distributions many times.
- Compute the empirical returns distribution of the portfolio.
- Compute the (e.g., 1%, 5%) quantile.

Monte Carlo Simulation Advantages

- Does not assume any specific models, or forms of distributions.
- Can incorporate any information, even subjective views.
- Can do scenario tests by twisting the parameters.
  - sensitivity analysis of parameters
  - give more weightings to more recent data
- Can work with non-linear assets, e.g., options.
- Can track path-dependence.

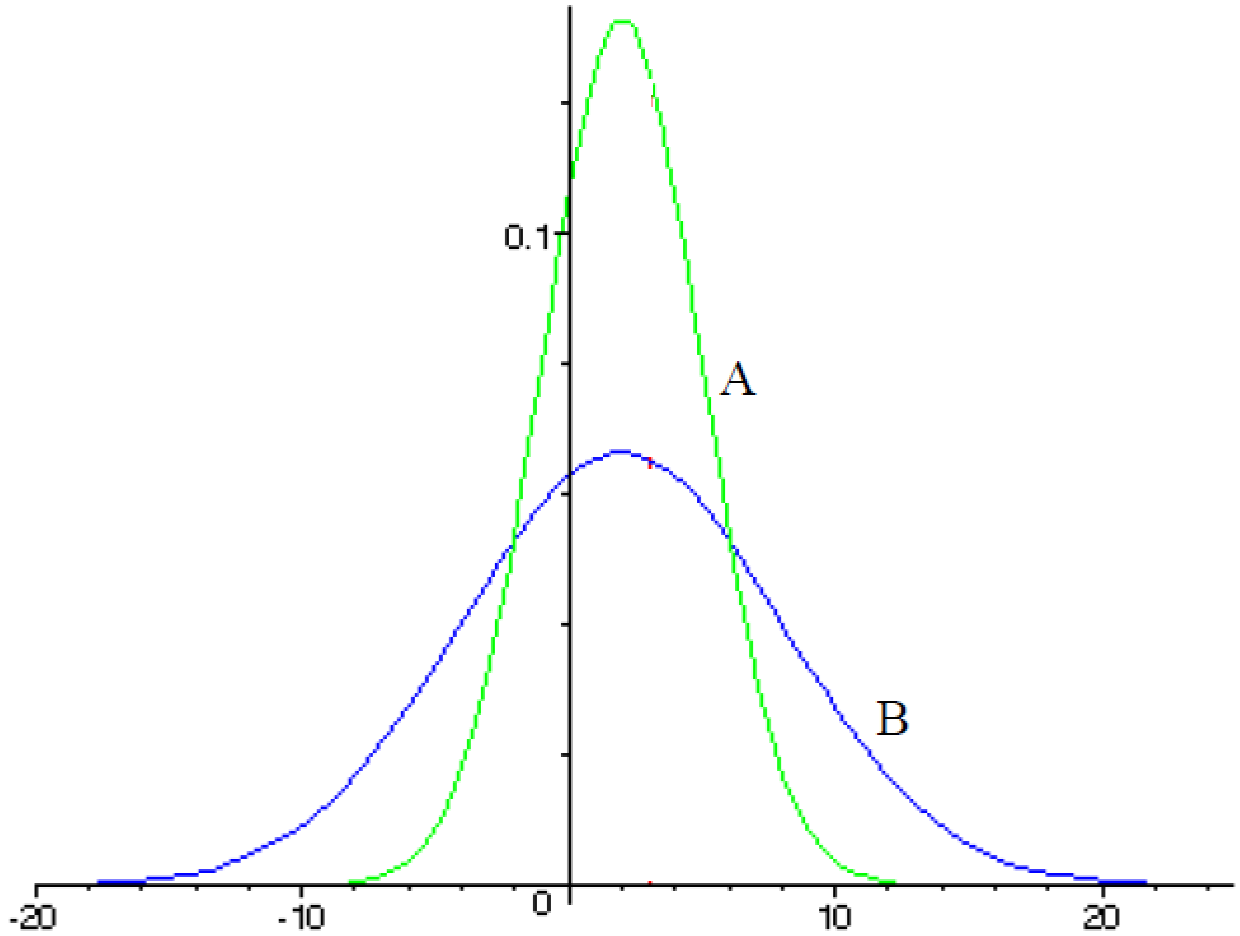
Monte Carlo Simulation Disadvantages

- Slow.
  - To increase the precision by a factor of 10, we must make 100 times more simulations.
  - Various variance reduction techniques apply.
    - antithetic variates
    - control variates
    - importance sampling
    - stratified sampling
- Difficult to build a (high) multi-dimensional joint distribution from data.

100-Year Market Crash

- How do we incorporate rare events into our returns distributions, hence enhanced risk management?
- Statistics works very well when you have a large amount of data.
- How do we analyze for (very) small samples?

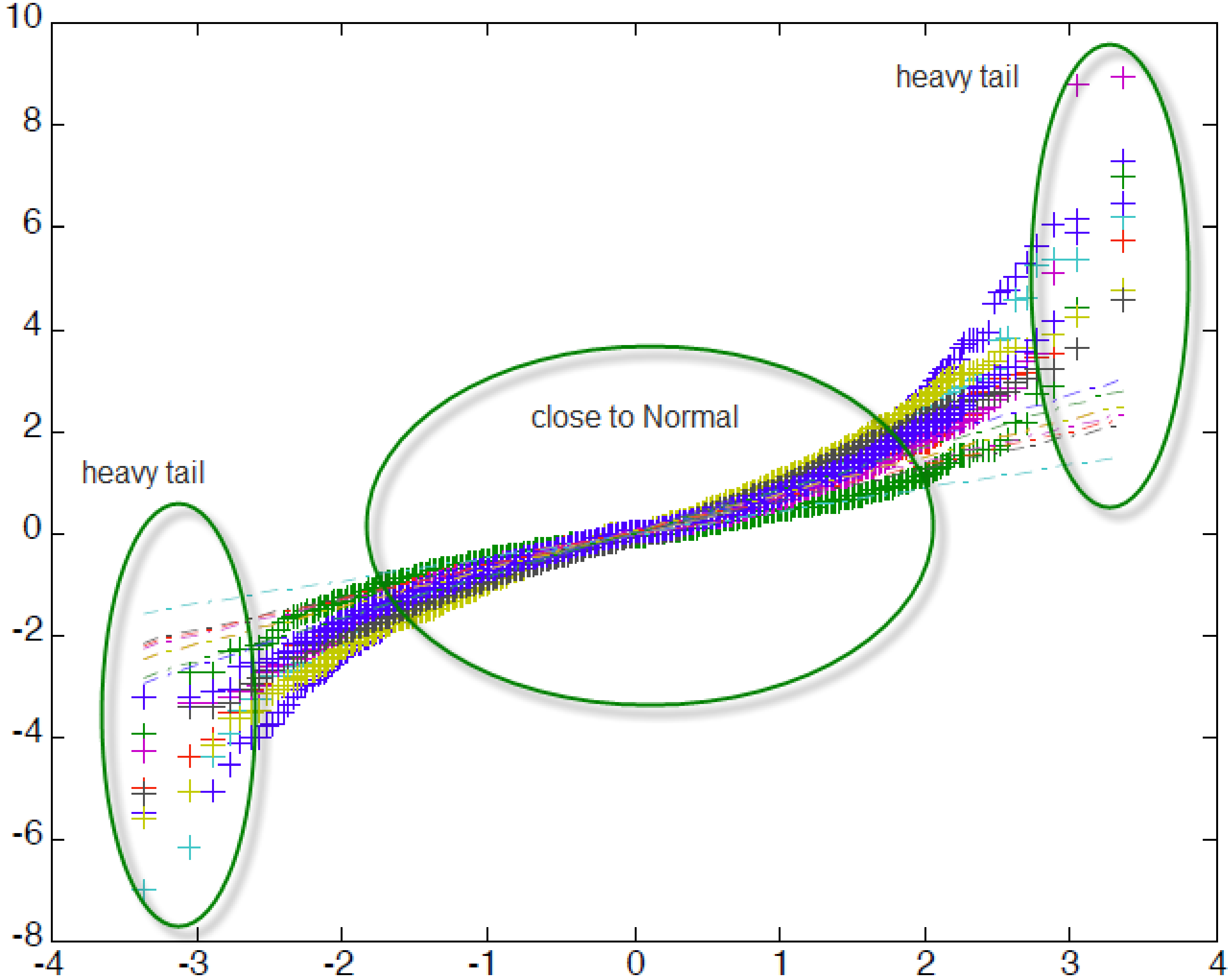
Fat Tails



QQ

- A QQ plots display the quintiles of the sample data against those of a standard normal distribution.
- This is the first diagnostic tool in determining whether the data have fat tails.

QQ Plot



heavy tail
close to Normal
heavy tail
10
8
6
4
2
0
-2
-4
-6
-8
-4
-3
-2
-1
0
1
2
3
4

Asymptotic Properties

- The (normalized) mean of a the sample mean of a large population is normally distributed, *regardless of the generating distribution*.
- What about the sample maximum?

Intuition

- Let $X_1, \dots, X_n$ be i.i.d. with distribution $F(x)$.
- Let the sample maxima be $M_n = X_{(n)} = \max_i X_i$.
- $P(M_n \leq x) = P(X_1 \leq x, \dots, X_n \leq x)$
- $= \prod_{i=1}^{n} P(X_i \leq x) = F^n(x)$
- What is $\lim_{n \to \infty} F^n(x)$?

Convergence

- Suppose we can scale the maximums $\{c_n\}$ and change the locations (means) $\{d_n\}$.
- There may exist non-negative sequences of these such that
  - ${c_n}^{-1}(M_n - d_n) \to Y$, Y is not a point
  - $H(x) = \lim_{n\to\infty} P({c_n}^{-1}(M_n - d_n) \leq x)$
  - $= \lim_{n\to\infty} P(M_n \leq c_n x + d_n)$
  - $= \lim_{n\to\infty} F^n(c_n x + d_n)$

Example 1 (Gumbel)

- $F(x) = 1 - e^{-\lambda x}, x > 0.$
- Let $c_n = \lambda^{-1}, d_n = \lambda^{-1}\log n.$
- $P(\lambda(M_n - \lambda^{-1}\log n) \leq x)$
- $= P(M_n \leq \lambda^{-1}(x + \log n))$
- $= \left(1 - e^{-(x+\log n)}\right)^n$
- $= \left(1 - \frac{e^{-x}}{n}\right)^n$
- $\rightarrow e^{-e^{-x}} = e^{-e^{-x}} 1_{\{x>0\}}$

Example 2 (Fre´chet)

- $F(x) = 1 - \frac{\theta^\alpha}{(\theta+x)^\alpha} = 1 - \frac{1}{\left(1+\frac{x}{\theta}\right)^\alpha}, x > 0.$
- Let $c_n = \theta n^{\frac{1}{\alpha}}, d_n = 0.$
- $P\left(\vartheta^{-1} n^{-1/\alpha} M_n \leq x\right)$
- $= P\left(M_n \leq \vartheta n^{1/\alpha} x\right)$
- $= \left(1 - \frac{1}{\left(1+n^{1/\alpha}x\right)^\alpha}\right)^n \sim \left(1 - \frac{1}{\left(n^{1/\alpha}x\right)^\alpha}\right)^n$
- $= \left(1 - \frac{x^{-\alpha}}{n}\right)^n$
- $\rightarrow e^{-x^{-\alpha}} 1_{\{x>0\}}$

Fisher-Tippett Theorem

- It turns out that H can take only one of the three possible forms.
- Fre´chet
  - $\Phi_\alpha(x) = e^{-x^{-\alpha}} 1_{\{x>0\}}$
- Gumbel
  - $\Lambda(x) = e^{-e^{-x}} 1_{\{x>0\}}$
- Weibull
  - $\Psi_\alpha(x) = e^{-(-x)^\alpha} 1_{\{x<0\}}$

Maximum Domain of Attraction

- Fre´chet
  - Fat tails
  - E.g., Pareto, Cauchy, student t,
- Gumbel
  - The tail decay exponentially with all finite moments.
  - E.g., normal, log normal, gamma, exponential
- Weibull
  - Thin tailed distributions with finite upper endpoints, hence bounded maximums.
  - E.g., uniform distribution

Why Fre´chet?

- Since we care about fat tailed distributions for financial asset returns, we rule out Gumbel.
- Since financial asset returns are theoretically unbounded, we rule out Weibull.
- So, we are left with Fre´chet, the most common MDA used in modeling extreme risk.

Fre´chet Shape Parameter

- α is the shape parameter.
- Moments of order r greater than α are infinite.
- Moments of order r smaller than α are finite.
  - Student t distribution has $\alpha \geq 2$. So its mean and variance are well defined.

Fre´chet MDA Theorem

- $F \in$ MDA H, H Fre´chet if and only if
- the complement cdf $\bar{F}(x) = x^{-\alpha} L(x)$
- L is slowly varying function
  - $\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1,\ t > 0$
- This restricts the maximum domain of attraction of the Fre´chet distribution quite a lot, it consists only of what we would call heavy tailed distributions.

Generalized Extreme Value Distribution (GEV)

- $H_\tau(x) = e^{-(1+\tau x)^{-\frac{1}{\tau}}}, \tau \neq 0$
- $H_\tau(x) = e^{-e^{-x}}, \tau = 0$
  - $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^{-n} = e^{-x}$
- tail index $\tau = \frac{1}{\alpha}$
- Fre′chet: $\tau > 0$
- Gumbel:$\tau = 0$
- Weibull: $\tau < 0$

Generalized Pareto Distribution

- $G_\tau(x) = 1 - (1 + \tau x)^{-\frac{1}{\tau}}$
- $G_0(x) = 1 - e^{-x}$
  - simply an exponential distribution
- Let $Y = \beta X$, $X \sim G_\tau$.
- $G_{\tau,\beta} = 1 - \left(1 + \tau \frac{y}{\beta}\right)^{-\frac{1}{\tau}}$
- $G_{0,\beta} = 1 - e^{-\frac{y}{\beta}}$

The Excess Function

- Let u be a tail cutoff threshold.
- The excess function is defined as:
  - $F_u(x) = 1 - \bar{F}_u(x)$
  - $\bar{F}_u(x) = P(X - u > x | X > u) = \frac{P(X>u+x)}{P(X>u)} = \frac{\bar{F}(x+u)}{\bar{F}(u)}$

Asymptotic Property of Excess Function

- Let $x_F = \inf\{x: F(x) = 1\}$.
- For each τ, $F \in \mathrm{MDA}(H_\tau)$, if and only if
  - $\lim_{u \to x_F^-} \sup_{0<x<x_F-u} \left| F_u(x) - G_{\tau,\beta(u)}(x) \right| = 0$
- If $x_F = \infty$, we have
  - $\lim_{u \to \infty} \sup_{x} \left| F_u(x) - G_{\tau,\beta(u)}(x) \right| = 0$
- Applications: to determine τ, u, etc.

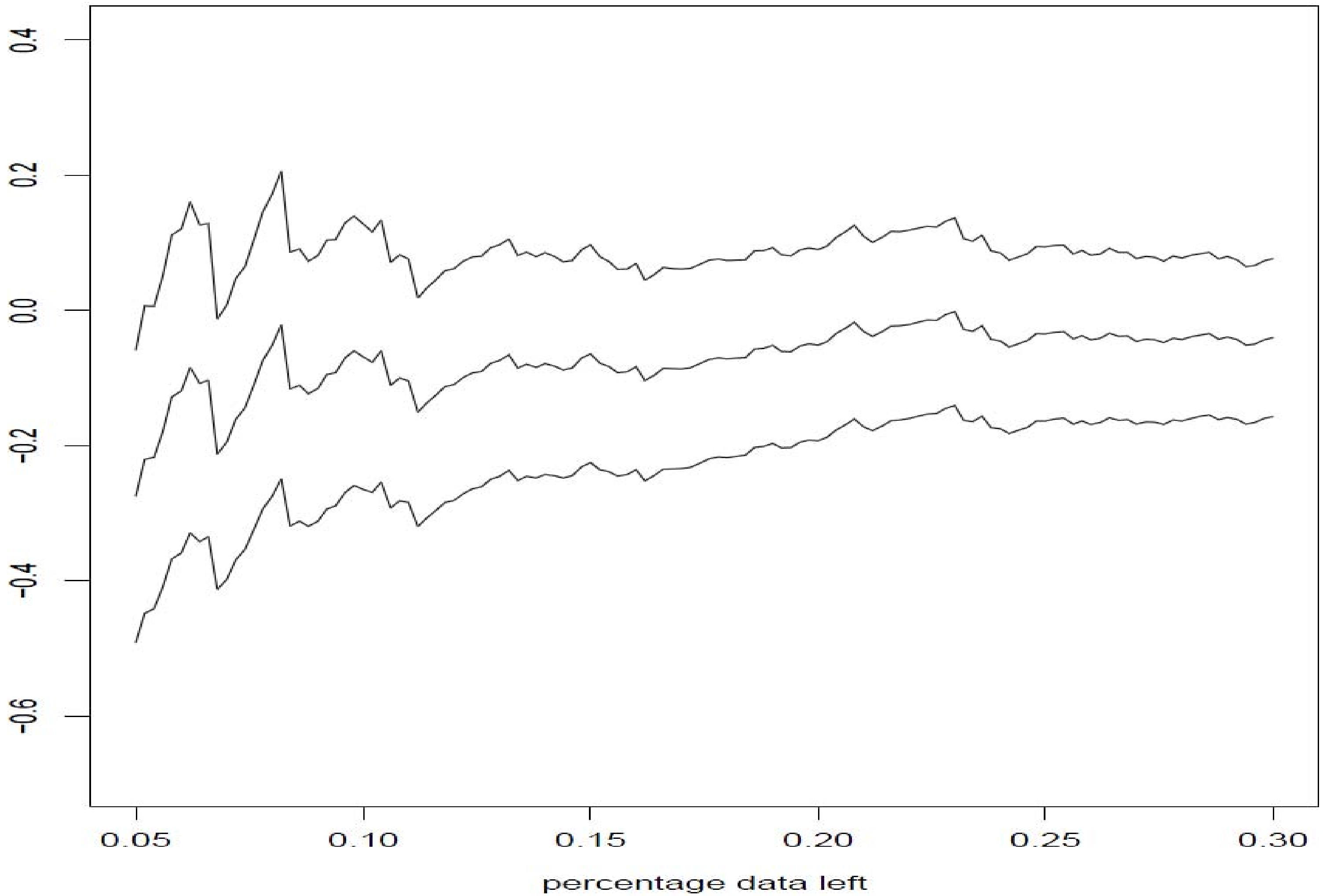
Tail Index Estimation by Quantiles

- Hill, 1975
- Pickands, 1975
- Dekkers and DeHaan, 1990

Hill Estimator

- $\tau_{n,m}{}^{H} = \frac{1}{m-1} \sum_{i=1}^{m-1} \left(\ln X^{*}{}_{i} - \ln X^{*}{}_{n-m,n} \right)$
- X^*: the order statistics of observations
- m: the number of observations in the (left) tail
- Mason (1982) shows that $\tau_{n,m}{}^{H}$ is a consistent estimator, hence convergence to the true value.
- Pictet, Dacorogna, and Muller (1996) show that in finite samples the expectation of the Hill estimator is biased.
- In general, bigger (smaller) m gives more (less) biased estimator but smaller (bigger) variance.

POT Plot



Pickands Estimator

- $$\tau_{n,m}{}^{P} = \frac{\ln(X^*{}_m - X^*{}_{2m})/(X^*{}_{2m} - X^*{}_{4m})}{\ln 2}$$

Dekkers and DeHaan Estimator

- $\tau_{n,m}{}^{D} = \tau_{n,m}{}^{H} + 1 - \frac{1}{2}\left(1 - \frac{\left(\tau_{n,m}{}^{H}\right)^2}{\tau_{n,m}{}^{H2}}\right)^{-1}$
- $\tau_{n,m}{}^{H2} = \frac{1}{m-1}\sum_{i=1}^{m-1}(\ln X^*{}_i - \ln X^*{}_m)^2$

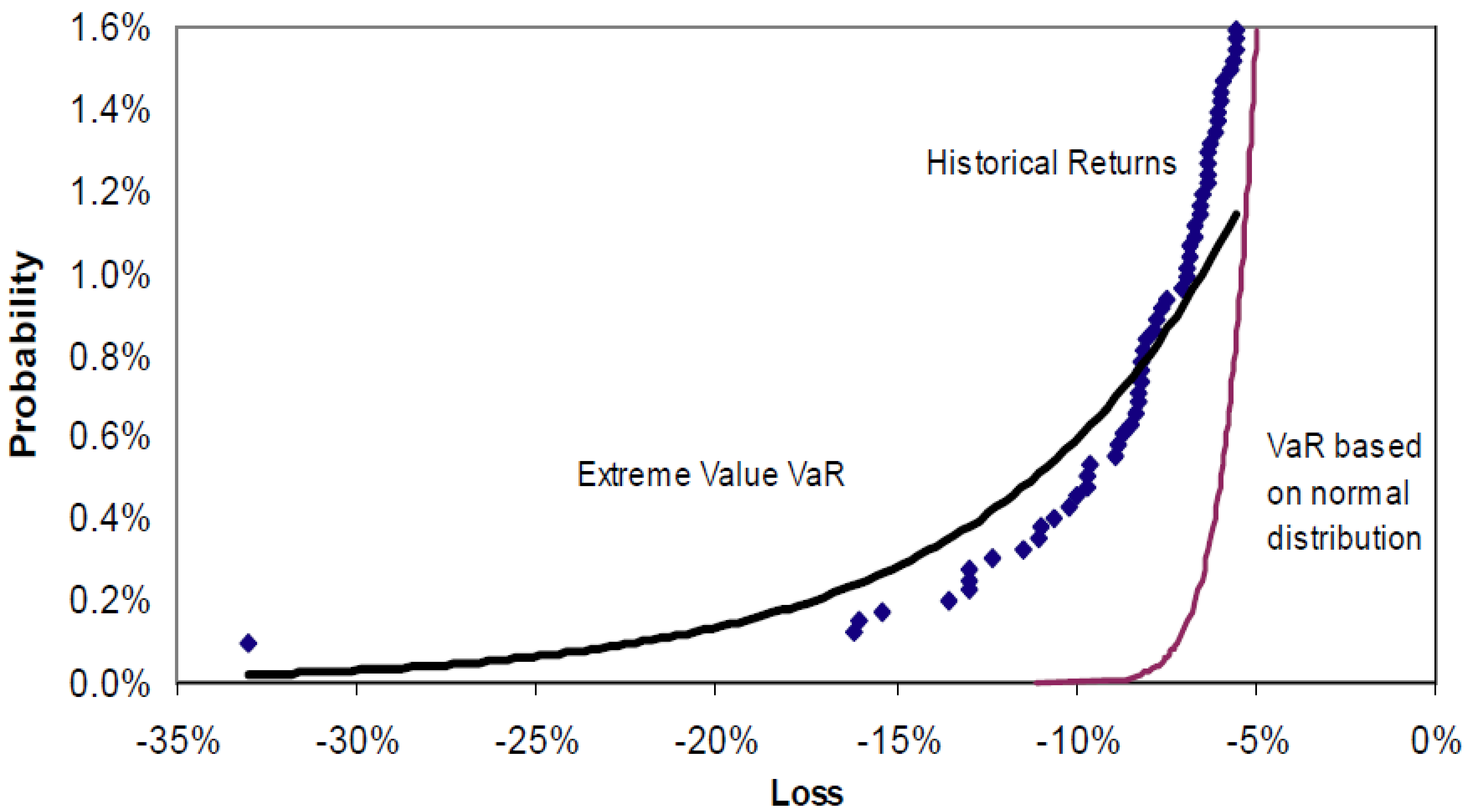
VaR using EVT

- For a given probability $q > F(u)$ the VaR estimate is calculated by inverting the excess function. We have:
- $\widehat{\mathrm{VaR}_q} = u + \frac{\hat{\beta}}{\hat{\tau}}\left(\left(\frac{n}{m}(1-q)\right)^{-\hat{\tau}} - 1\right)$
- Confidence interval can be computed using profile likelihood.

ES using EVT

- $\widehat{\mathrm{ES}}_q = \dfrac{\widehat{\mathrm{VaR}}_q}{1-\hat{\tau}} + \dfrac{\hat{\beta}-\hat{\tau}u}{1-\hat{\tau}}$

VaR Comparison



- http://www.fea.com/resources/pdf/a_evt_1.pdf