Introduction to Algorithmic Trading Strategies
Lecture 3

Trend Following

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References

- Introduction to Stochastic Calculus with Applications. Fima C Klebaner. 2nd Edition.
- Estimating continuous time transition matrices from discretely observed data. Inamura, Yasunari. April 2006.
- Optimal Trend Following Trading Rules. Dai, Min and Zhang, Qing and Zhu, Qiji Jim. 2011.

Stochastic Calculus

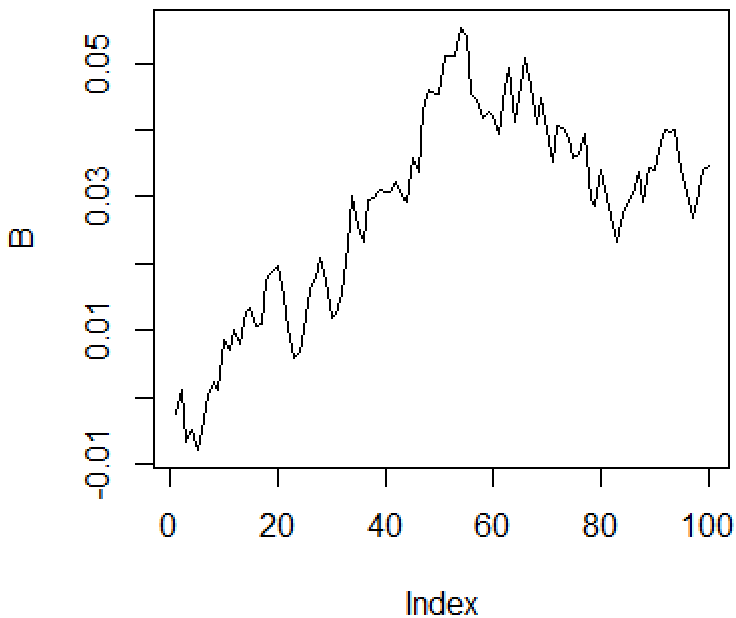
Brownian Motion

- Independence of increments.
  - $dB = B(t) - B(s)$ is independent of any history up to s.
- Normality of increments.
  - dB is normally distributed with mean 0 and variance $t - s$.
- Continuity of paths.
  - $B(t)$ is a continuous function of t.

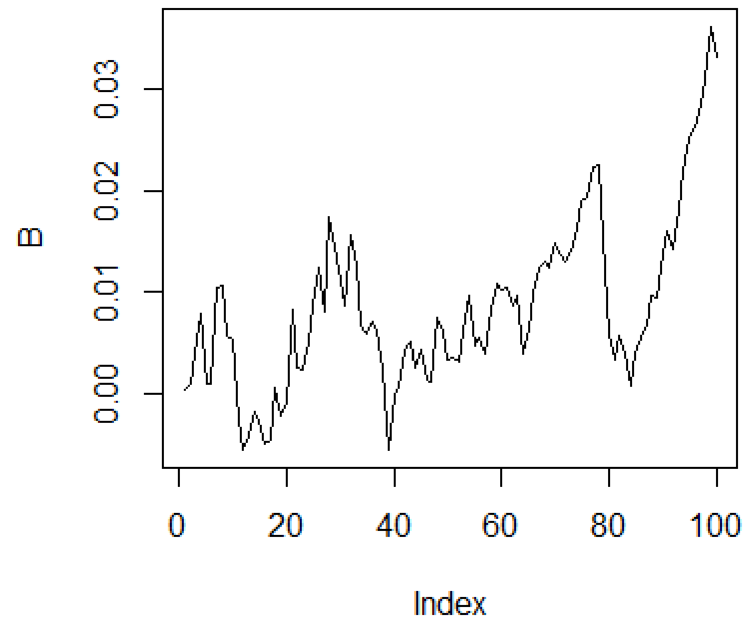
Stochastic vs. Newtonian Calculus

- Newtonian calculus: when you zoom in a function enough, the little segment looks like a straight line, hence the approximation:
  - $dy = \dot{f}\,dt$
  - Derivative exists.
  - The function is differentiable.
- Stochastic calculus: no matter how much you zoom in or how small dt is, the function still looks very zig-zag and random. It is nothing like a straight line.
  - For Brownian motion, however much you zoom in, how small the segment is, it still just looks very much like a Brownian motion.
  - The function is therefore no where differentiable.

Examples



not a trend



not mean reversion

dBdB

- $(dB)^2 = dBdB = dt$
- $\int_0^t (dB_t)^2 \approx \sum_{i=1}^{n} {Z_{n,i}}^2$
  - $Z_{n,i}$ is of $N\left(0, \frac{t}{n}\right)$ for all i.
- $\int_0^t (dB_t)^2 \approx$ sum of variances of $Z_{n,i} \approx t$
- Making $n \to \infty$ or $dt \to 0$, we have
  - $\int_0^t (dB_t)^2 = t$, convergence in probability
  - $(dB_t)^2 = dt$, in differential form
- $dBdt = 0$
- $dtdt = 0$

Asset Price Model

- Want to model asset price movement.
- Change in price = dS
  - Change in price is not too meaningful as \$1 change in a penny stock is more significant than \$1 change in GOOG.
- Return = $\frac{dS}{S}$
- Model return using two parts.
  - Deterministic: μdt, the predictable part. E.g., fixed deposit interest rate.
  - Random/stochastic: σdB, where σ is the volatility of returns and dB is a sample from a probability distribution, e.g., Normal.

Geometric Brownian Motion

- Asset price: $\frac{dS}{S} = \mu dt + \sigma dB$
  - dB: normally distributed
  - $\mathrm{E}(dB) = 0$
  - Variance = dt. It is intuitive that dB should be scaled by dt otherwise the (random) return drawn would be too big from any Normal distribution for $dt \to 0$.
  - $dB = \phi\sqrt{dt}$, where ϕ is a standard Normal distribution.
- Reasonably good model for stocks and indices.
  - Real data have more big rises and falls than this model predicts, i.e., extreme events.

GBM Properties

- Markov property: the distribution of the next price $S + dS$ depend only on the current price S.
- $\mathrm{E}(dS) = \mathrm{E}(\mu S dt + \sigma S dB)$
  - $= \mathrm{E}(\mu S dt) + \mathrm{E}(\sigma S dB)$
  - $= \mu S dt + \sigma S\, \mathrm{E}(dB)$
  - $= \mu S dt$
- $\mathrm{Var}(dS) = \mathrm{E}\left(dS^2\right) - \mathrm{E}(dS)^2 = \sigma^2 S^2 dt$
  - $dBdt = 0$
  - $dtdt = 0$
  - $dBdB = dt$

Stochastic Differential Equation

- Both μ and σ can be as simple as constants, deterministic functions of t and S, or as complicated as stochastic functions adapted to the filtration generated by $\{S_t\}$.
- $dS_t = \mu dt + \sigma dB_t$
- $dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t$

Univariate Ito's Lemma

- Assume
  - $dX_t = \mu_t dt + \sigma_t dB_t$
  - $f(t, X_t)$ is twice differentiable of two real variables
- We have
  - $df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{{\sigma_t}^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x} dB_t$

"Proof"

- Taylor series
- $df(t,X)$
- $= f_t dt + f_X dX + \frac{1}{2}(f_{tt} dt dt + f_{tX} dt dX + f_{Xt} dX dt + f_{XX} dX dX)$
- $= f_t dt + f_X dX + \frac{1}{2} f_{XX} dX dX$
- $= f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX}(\mu dt + \sigma dB)^2$
- $= f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX}(\mu dt + \sigma dB)^2$
- $= f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX}\left(\mu^2 dt^2 + \sigma^2 dB^2 + 2\mu dt \sigma dB\right)$
- $= f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX} \sigma^2 dB^2$
- $= f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX} \sigma^2 dt$
- $= \left(f_t + \mu f_X + \frac{1}{2}\sigma^2 f_{XX}\right) dt + \sigma f_X dB$

Log Example

- For G.B.M., $dX_t = \mu X_t dt + \sigma X_t dz_t$, $d \log X_t = ?$
- $f(x) = \log(x)$
- $\frac{\partial f}{\partial t} = 0$
- $\frac{\partial f}{\partial x} = \frac{1}{x}$
- $\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$
- $d \log X_t = \left(\mu X_t \frac{1}{X_t} + \frac{(\sigma X_t)^2}{2}\left(-\frac{1}{{X_t}^2}\right)\right) dt + \sigma X_t \left(\frac{1}{X_t}\right) dB_t$
- $= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB_t$

Exp Example

- $dX_t = \mu dt + \sigma dz_t, de^{X_t} =?$
- $f(x) = e^x$
- $\frac{\partial f}{\partial t} = 0$
- $\frac{\partial f}{\partial x} = e^x$
- $\frac{\partial^2 f}{\partial x^2} = e^x$
- $de^{X_t} = \left(\mu e^{X_t} + \frac{\sigma^2}{2} e^{X_t}\right) dt + \sigma e^{X_t} dB_t$
- $= e^{X_t}\left[\left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma dB_t\right]$
- $\frac{de^{X_t}}{e^{X_t}} = \left(\mu + \frac{\sigma^2}{2}\right) dt + \sigma dB_t$

Stopping Time

- A stopping time is a random time that some event is known to have occurred.
- Examples:
  - when a stock hits a target price
  - when you run out of money
  - when a moving average crossover occurs
- When a stock has bottomed out is NOT a stopping time because you cannot tell when it has reached its minimum without future information.

Optimal Trend Following Strategy

Asset Model

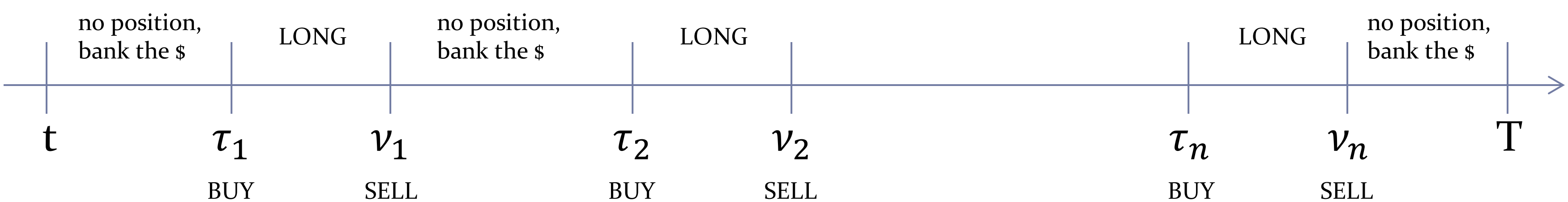
- Two state Markov model for a stock's prices: BULL and BEAR.
- $dS_r = S_r\left[\mu_{\alpha_r} dr + \sigma dB_r\right], t \leq r \leq T < \infty$
  - The trading period is between time $[t, T]$.
- $\alpha_r = \{1,2\}$ are the two Markov states that indicates the BULL and BEAR markets.
  - $\mu_1 > 0$
  - $\mu_2 < 0$
- $Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$, the generator matrix for the Markov chain.

Generator Matrix

- From transition matrix to generator matrix:
  - $P(t, t+\Delta t) = I + \Delta t Q + o(\Delta t)$
- We can estimate Q by estimating P.
  - $P(t, s) = P(t, t + m\Delta t)$
  - $\approx (I + \Delta t Q)^m = \left(I + \frac{(s-t)}{m} Q\right)^m = \exp\big((s-t)Q\big)$
  - $P(t, t+\Delta t) = \exp(\Delta t Q)$
- $\hat{P}(t) \approx \exp\big(\Delta t \hat{Q}\big)$
- $\hat{Q} \approx \frac{1}{\Delta t} \log \hat{P}(t)$
- E.g., $\Delta t = \frac{1}{252}$

Optimal Stopping Times

$i = 0, \Lambda_0 = \{\tau_1, \nu_1, \tau_2, \nu_2, \cdots\}$



$i = 0, \Lambda_0 = \{\nu_1, \tau_2, \nu_2, \tau_3, \cdots\}$

Parameters

- $\rho \geq 0$, interest free rate
- $S_t = S$, the initial stock price
- $i = \{0,1\}$, the net position
- K_b, the transaction cost for BUYs in percentage
- K_s, the transaction cost for SELLs in percentage

Expected Return (starting flat)

- When $i = 0$,
- $\mathrm{E}_{0,t}(R_t) =$

$$\mathrm{E}_t\left(e^{\rho(\tau_1 - t)} \prod_{n=1}^{\infty} \frac{S_{\nu_n}}{S_{\tau_n}} \left[\frac{1-K_s}{1+K_b}\right]^{I_{\{\tau_n < T\}}} e^{\rho(\tau_{n+1} - \nu_n)}\right)$$

  - You are long between τ_n and ν_n and the return is determined by the price change discounted by the commissions.
  - You are flat between ν_n and τ_{n+1} and the money grows at the risk free rate.

Expected Return (starting long)

- When $i = 1$,
- $\mathrm{E}_{1,t}(R_t) =$
  $$\mathrm{E}_t\left(\left[\frac{S_{\nu_1}}{S} e^{\rho(\tau_2-\nu_1)}(1-K_s)\right]\prod_{n=2}^{\infty}\frac{S_{\nu_n}}{S_{\tau_n}}\left[\frac{1-K_s}{1+K_b}\right]^{I_{\{\tau_n<T\}}} e^{\rho(\tau_{n+1}-\upsilon_n)}\right)$$
  - You sell the long position at ν_1.

Value Functions

- It is easier to work with the log of the returns.
- $J_0\,(S,\alpha,t,\Lambda_0) = \mathrm{E}_t\left(\rho(\tau_1 - t) + \sum_{n=1}^{\infty}\left\{\log\frac{S_{\nu_n}}{S_{\tau_n}} + I_{\{\tau_n<T\}}\log\frac{1-K_s}{1+K_b} + \rho(\tau_{n+1} - \upsilon_n)\right\}\right)$
- $J_1\,(S,\alpha,t,\Lambda_1) =$
  $\mathrm{E}_t\left(\left[\log\frac{S_{\nu_1}}{S} + \rho(\tau_2 - \nu_1) + \log(1 - K_s)\right] + \sum_{n=2}^{\infty}\left\{\log\frac{S_{\nu_n}}{S_{\tau_n}} + I_{\{\tau_n<T\}}\log\frac{1-K_s}{1+K_b} + \rho(\tau_{n+1} - \upsilon_n)\right\}\right)$

Conditional Probability of Bull Market

- $p_r = P(\alpha_r = 1 \mid \sigma\{S_u : 0 \ \le u \le r\})$
- p_r is therefore a random process driven by the same Brownian motion that drives S_u.
- $dp_r = [-(\lambda_1 + \lambda_2)p_r + \lambda_2]dr + \frac{(\mu_1 - \mu_2)p_r(1-p_r)}{\sigma} d\widehat{B_r}$
  - $d\widehat{B_r} = \frac{d\,\log(S_r) - \left[(\mu_1 - \mu_2)p_r + \mu_2 - \sigma^2/2\right]dr}{\sigma}$
- $p_{t+1} = p_t + g(p_t)\Delta t + \frac{(\mu_1 - \mu_2)p_t(1-p_t)}{\sigma^2} \log\left(\frac{S_{t+1}}{S_t}\right)$
  - $g(p) = -(\lambda_1 + \lambda_2)p + \lambda_2 - \frac{(\mu_1 - \mu_2)p(1-p)\left[(\mu_1 - \mu_2)p + \mu_2 - \sigma^2/2\right]}{\sigma^2}$
  - Make p_{t+1} between 0 and 1 if it falls outside the bounds.

Objective

- Find an optimal trading sequence (the stopping times) so that the value functions are maximized.
- $V_i(p,t) = \sup_{\Lambda_i} J_i(S,p,t,\Lambda_i)$
  - V_i: the maximum amount of expected returns

Realistic Expectation of Returns

- Lower bounds:
  - $V_0(p,t) \geq \rho(T-t)$
  - $V_1(p,t) \geq \rho(T-t) + \log(1-K_s)$
- Upper bound:
  - $V_i(p,t) \leq \left(\mu_1 - \frac{\sigma^2}{2}\right)(T-t)$
- No trading zones:
  - One should never buy when $\rho \geq \mu_1 - \frac{\sigma^2}{2}$

Principal of Optimality

- Given an optimal trading sequence, Λ_i, starting from time t, the truncated sequence will also be optimal for the same trading problem starting from any stopping times τ_n or υ_n.

Coupled Value Functions

$$\begin{cases} V_0(p,t) = \sup\limits_{\tau_1} E_t\{\rho(\tau_1 - t) - \log(1+K_b) + V_1\left(p_{\tau_1}, \tau_1\right)\} \\ V_1(p,t) = \sup\limits_{\nu_1} E_t\left\{\log\frac{S_{\nu_1}}{S_t} + \log(1-K_s) + V_0\left(p_{\nu_1}, \nu_1\right)\right\} \end{cases}$$

Hamilton-Jacobi-Bellman Equations

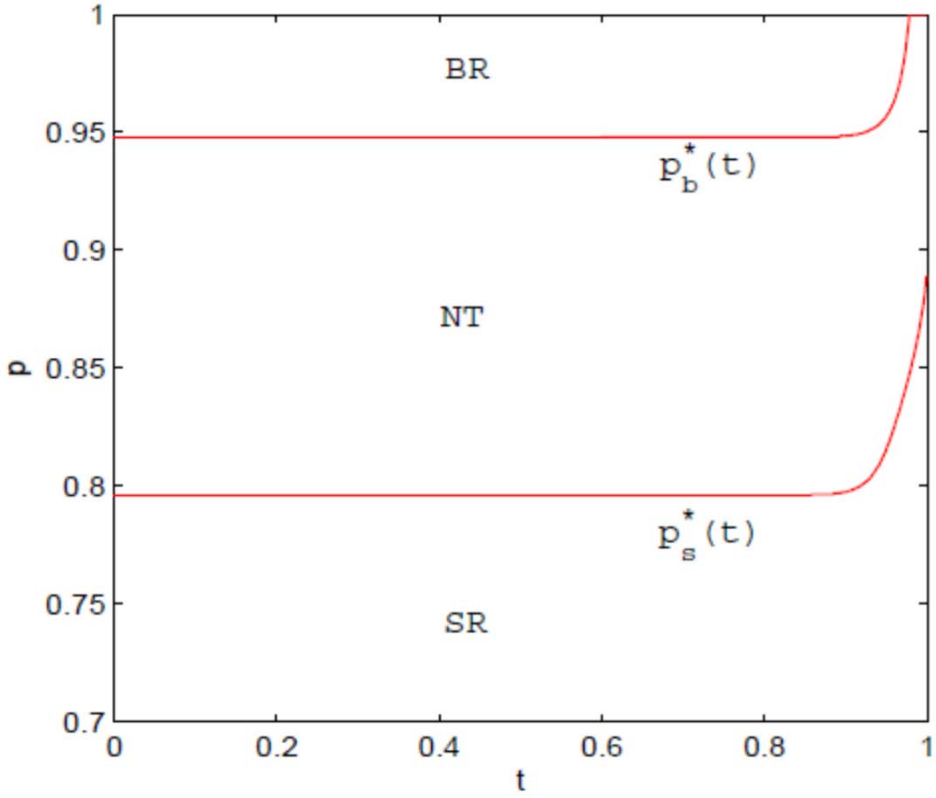
- $\begin{cases} \min\{-\mathcal{L}V_0 - \rho, V_0 - V_1 + \log(1+K_b)\} = 0 \\ \min\{-\mathcal{L}V_1 - f(\rho), V_1 - V_0 - \log(1-K_s)\} = 0 \end{cases}$
- with terminal conditions: $\begin{cases} V_0(p,T) = 0 \\ V_1(p,T) = \log(1-K_s) \end{cases}$
- $\mathcal{L} = \partial_t + \frac{1}{2}\left(\frac{(\mu_1-\mu_2)p(1-p)}{\sigma}\right)^2 \partial_{pp} + [-(\lambda_1+\lambda_2)p + \lambda_2]\partial_p$

Penalty Formulation Approach

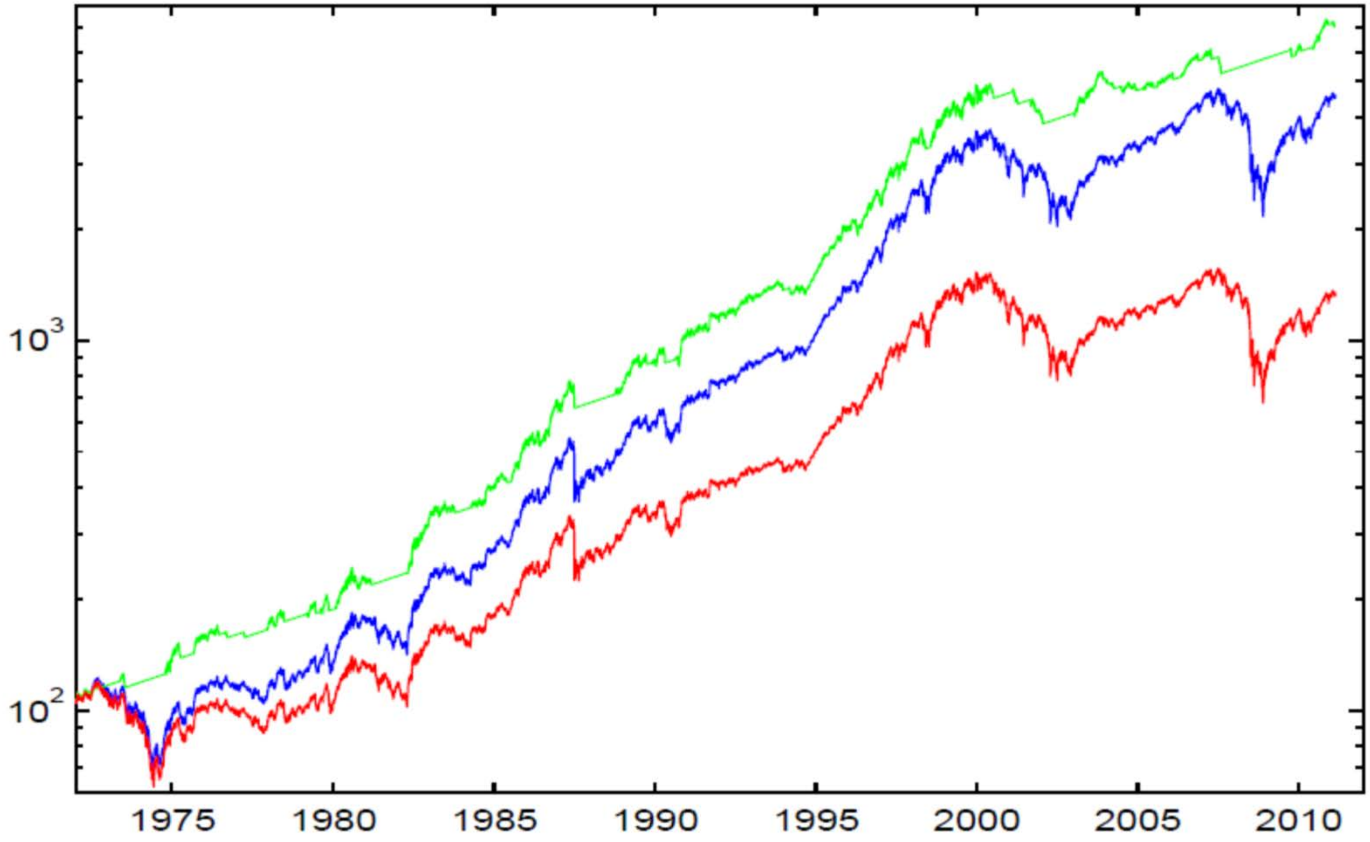
- The HJB formulation is equivalent to this system of PDEs.
- $$\begin{cases} -\mathcal{L}V_0 - h(\rho) = \hat{\xi}(V_1 - V_0 - \log(1 + K_b))^+ \\ -\mathcal{L}V_1 - f(\rho) = \hat{\xi}(V_0 - V_1 + \log(1 - K_s))^+ \end{cases}$$
- $\hat{\xi}$ is a penalization factor such that $\hat{\xi} \to \infty$.
- $h(\rho) = \rho$

Trading Boundaries

- $BR = \{p \in (0,1) \times [0,T) : p \geq p_b^*(t)\}$
- $SR = \{p \in (0,1) \times [0,T) : p \leq p_s^*(t)\}$

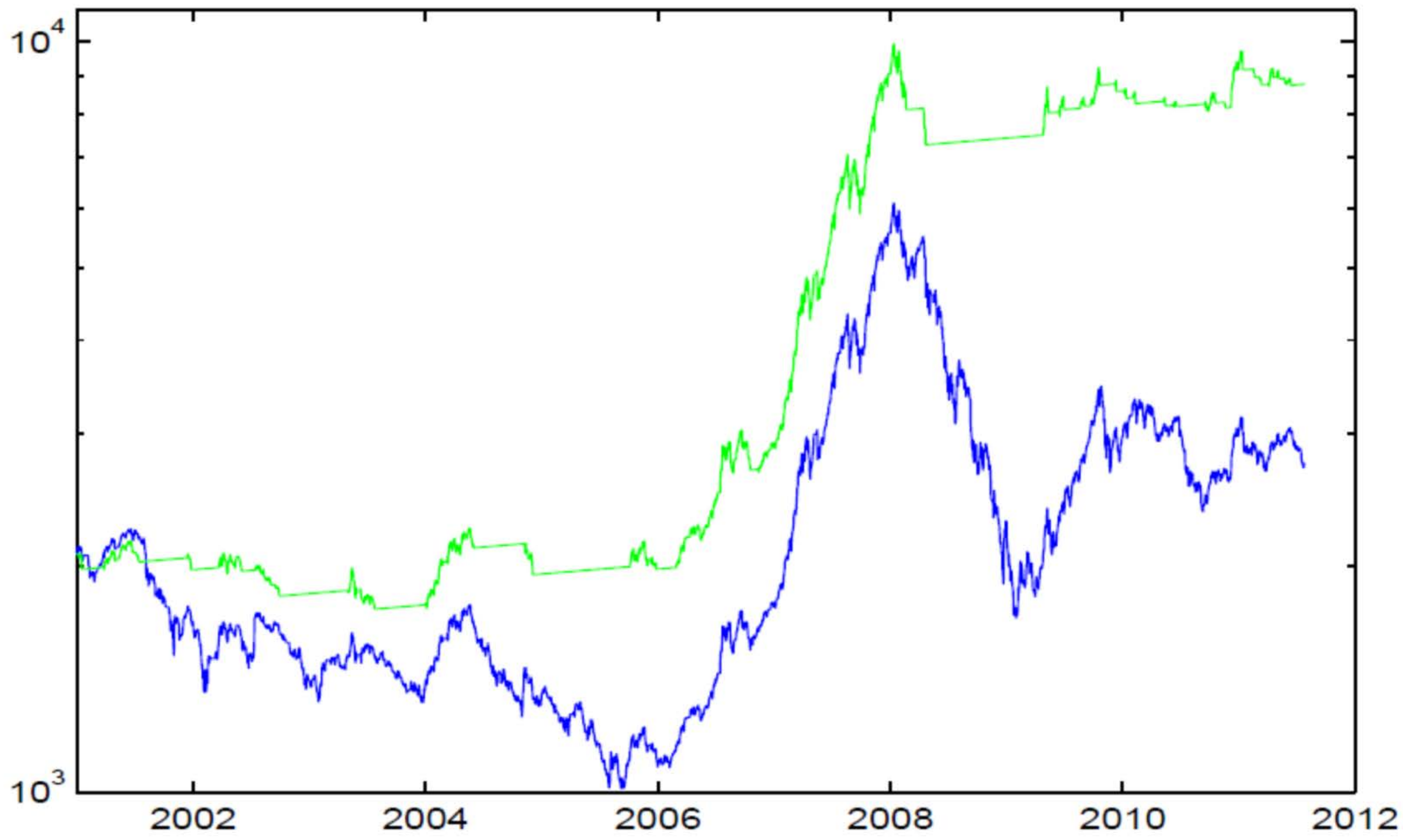


SP500



Trend following trading of SP500 1972–2011 compared with buy and hold

SSE



Trend following trading of SSE 2001–2011 compared with buy and hold

Problems

- The buy entry signals are always delayed (as expected).
- The market therefore needs to bull long enough for the strategy to make profit.
- If the bear market comes in too fast and hard, the exit entry may not come in soon enough.
  - A proper stoploss from max drawdown may help.

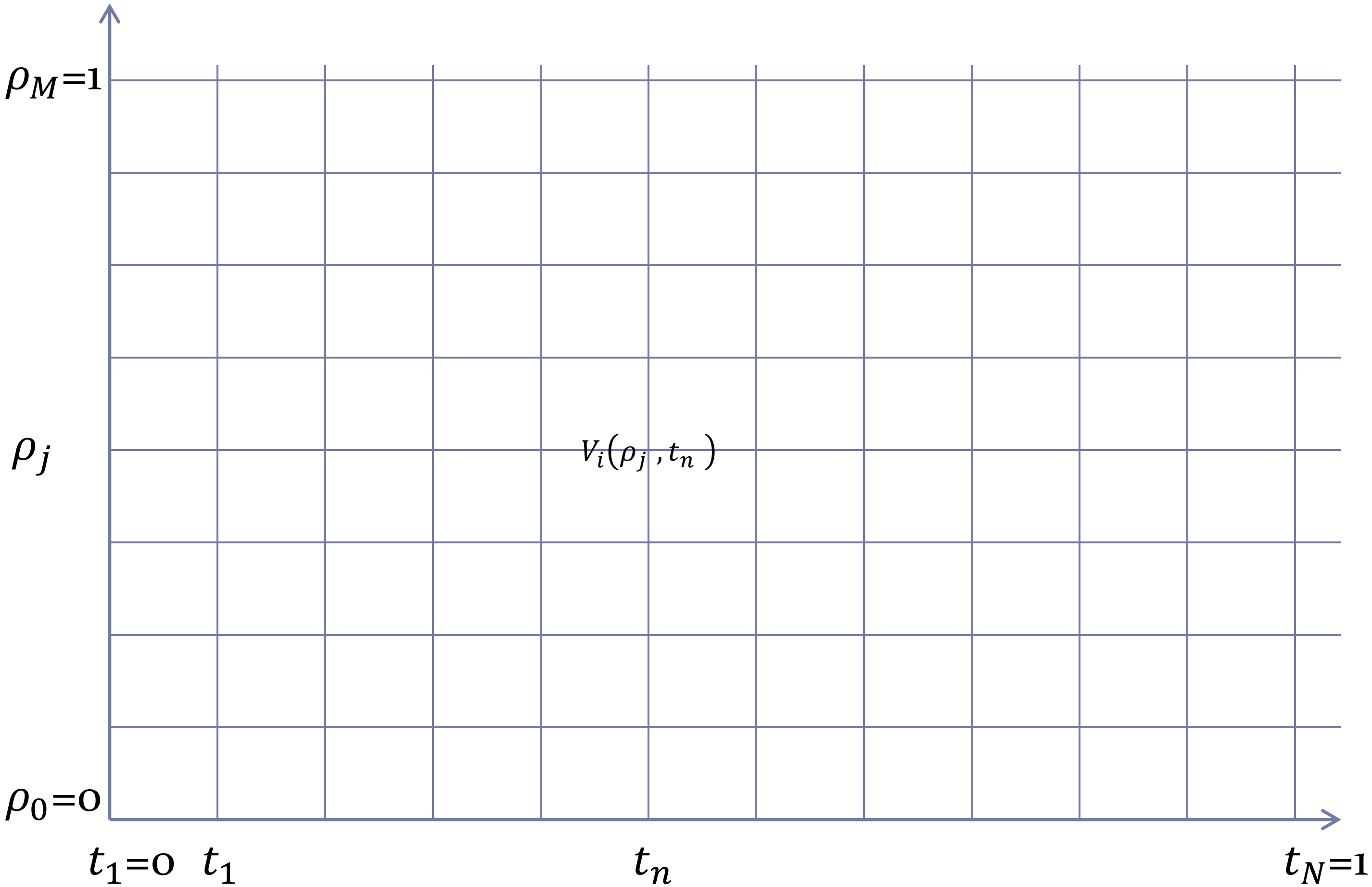
Finding the Boundaries

- At any time t, we need to find p_b^* and p_s^* to determine whether we buy or sell or go flat.
- p_b^* is determined by
  - $V_1(p_b^*, t) - V_0(p_b^*, t) = \log(1 + K_b)$
- p_s^* is determined by
  - $V_1(p_s^*, t) - V_0(p_s^*, t) = \log(1 - K_s)$
- Need to compute $V_0(p, t)$ and $V_1(p, t)$ for all p and t.
  - Solve the coupled PDEs.

Discretization of $\mathcal{L}$

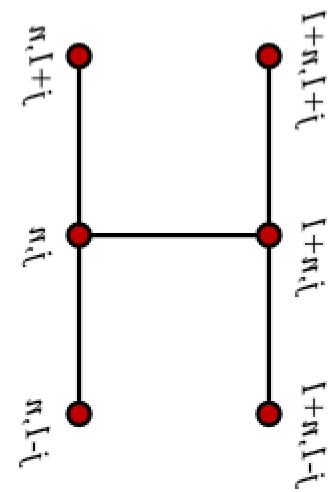
- $(\rho, t) \in [0,1] \times [0,1)$
- (ρ, t) as $\{(\rho_j, t_n)\}$
  - $j \in \{0, \cdots, M\}$
    - $\rho_0 = 0$
    - $\rho_M = 0$
  - $n \in \{1, \cdots, N\}$
    - $t_1 = 0$
    - $t_N = 1^-$
- $\frac{\partial V_i}{\partial t} \approx \frac{V_{i,j}^{n+1} - V_{i,j}^{n}}{\Delta t}$

The Grid



Crank-Nicolson Scheme

- $$\frac{\partial V_i}{\partial \rho} \approx \frac{1}{2}\left(\frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2\Delta\rho} + \frac{V_{i,j+1}^{n} - V_{i,j-1}^{n}}{2\Delta\rho}\right)$$



- $$\frac{\partial^2 V_i}{\partial \rho^2} \approx \frac{1}{2}\left(\frac{V_{i,j+1}^{n+1} - 2V_{i,j}^{n+1} + V_{i,j-1}^{n+1}}{(\Delta\rho)^2} + \frac{V_{i,j+1}^{n} - 2V_{i,j}^{n} + V_{i,j-1}^{n}}{(\Delta\rho)^2}\right)$$

RHS

- $\hat{\xi}(V_1 - V_0 - \log(1 + K_b))^+ = \hat{\xi}_{0,j}^{n+1/2}\left(\frac{V_{1,j}^{n+1} - V_{0,j}^{n+1}}{2} + \frac{V_{1,j}^{n} - V_{0,j}^{n}}{2} - k_b\right)$
  - $k_b = \log(1 + K_b)$
- $\hat{\xi}(V_0 - V_1 + \log(1 - K_s))^+ = \hat{\xi}_{1,j}^{n+1/2}\left(\frac{V_{0,j}^{n+1} - V_{1,j}^{n+1}}{2} + \frac{V_{0,j}^{n} - V_{1,j}^{n}}{2} + k_s\right)$
  - $k_s = \log(1 - K_s)$

Penalty

- $$\hat{\xi}_{0,j}^{n+1/2} = \begin{cases} \xi \Delta t, & \text{if } \frac{V_{1,j}^{n+1} - V_{0,j}^{n+1}}{2} + \frac{V_{1,j}^{n} - V_{0,j}^{n}}{2} - k_b > 0 \\ 0, & \text{otherwise} \end{cases}$$
- $$\hat{\xi}_{1,j}^{n+1/2} = \begin{cases} \xi \Delta t, & \text{if } \frac{V_{0,j}^{n+1} - V_{1,j}^{n+1}}{2} + \frac{V_{0,j}^{n} - V_{1,j}^{n}}{2} + k_s > 0 \\ 0, & \text{otherwise} \end{cases}$$

The Discretized PDEs

- $a_{1,j}V_{0,j+1}^{n+1} + a_{0,j}V_{0,j}^{n+1} + a_{-1,j}V_{0,j-1}^{n+1} = c_{0,j}^{n} + \hat{\xi}_{0,j}^{n+1/2}\left(\frac{V_{1,j}^{n+1}-V_{0,j}^{n+1}}{2} + \frac{V_{1,j}^{n}-V_{0,j}^{n}}{2} - k_b\right)$
- $a_{1,j}V_{1,j+1}^{n+1} + a_{0,j}V_{1,j}^{n+1} + a_{-1,j}V_{1,j-1}^{n+1} = c_{1,j}^{n} + \hat{\xi}_{1,j}^{n+1/2}\left(\frac{V_{0,j}^{n+1}-V_{1,j}^{n+1}}{2} + \frac{V_{0,j}^{n}-V_{1,j}^{n}}{2} + k_s\right)$
- $j \in \{1, \cdots, M-1\}$
  - $p = 0$, $p = 1$ are excluded.

The Coefficients

- $a_{-1,j} = -\frac{1}{2}\sigma_p^2 \frac{\Delta t}{2\Delta p^2} + \frac{1}{2}\mu_p \frac{\Delta t}{2\Delta p}$
- $a_{0,j} = 1 + \sigma_p^2 \frac{\Delta t}{2\Delta p^2}$
- $a_{1,j} = -\frac{1}{2}\sigma_p^2 \frac{\Delta t}{2\Delta p^2} - \frac{1}{2}\mu_p \frac{\Delta t}{2\Delta p}$
- $c_{0,j}^n = -a_{1,j}V_{0,j+1}^n + \left(2 - a_{0,j}\right)V_{0,j}^n - a_{-1,j}V_{0,j-1}^n + h(j\Delta p)\Delta t$
- $c_{1,j}^n = -a_{1,j}V_{1,j+1}^n + \left(2 - a_{0,j}\right)V_{1,j}^n - a_{-1,j}V_{1,j-1}^n + f(j\Delta p)\Delta t$

Boundary Conditions

- $j = 0, p = 0$
  - $a_{0,0}$=1
  - $a_{1,0} = 0$
  - $c_{0,0}^n = V_{0,1}^n + h(0)\Delta t$
  - $c_{1,0}^n = V_{1,1}^n + f(0)\Delta t$
- $j = M, p = 1$
  - $a_{0,M}$=1
  - $a_{-1,M} = 0$
  - $c_{0,M}^n = V_{0,M-1}^n + h(1)\Delta t$
  - $c_{1,M}^n = V_{1,M-1}^n + f(1)\Delta t$
- $n = 0,\ t = 0,$ for all $j \in \{0, \cdots, M\}$
  - $V_{0,j} = 0$
  - $V_{1,j} = k_s$

System in Matrix Form

- $\begin{cases} A^n \vec{V}_0^{n+1} = \vec{c}_0^{\,n} \\ A^n \vec{V}_1^{n+1} = \vec{c}_1^{\,n} \end{cases}$
- $A^n = \begin{bmatrix} a_{0,0} & 0 & 0 & \cdots & 0 \\ a_{-1,1} & a_{0,1} & a_{1,1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{-1,M-1} & a_{0,M-1} & a_{1,M-1} \\ 0 & \cdots & 0 & 0 & a_{0,M} \end{bmatrix}$

Solving the System of Equations

- We already know the values of $\vec{V}_0^0$ and $\vec{V}_1^0$.
- Starting from $n = 0$, using A^n, we iteratively solve for $\vec{V}_0^{n+1}$ and $\vec{V}_1^{n+1}$ until $n = M - 1$.
  - There are totally M systems of equations to solve.
- The equations are linear in the unknowns $V_{0,j}^{n+1}$ and $V_{1,j}^{n+1}$ if and only if $\hat{\xi}_{0,j}^{n+1/2} = 0$.
- When the equations are linear, we can use Thomas' algorithm to solve for a tri-diagonal system of linear equations.
- Otherwise, we use an iterative scheme.

Iterative Scheme

- 1st step, initialization, $k = 0$
  - Assume $\hat{\xi}_{i,j}^{n+1/2}(0) = 0$. Solve for $\vec{V}_i^{n+1}(0)$.
- kth step
  - Using $\vec{V}_i^{n+1}(k-1)$, update $\hat{\xi}_{i,j}^{n+1/2}(k)$.
  - When $\hat{\xi}_{i,j}^{n+1/2}(k) > 0$, adjust the $A^n(k)$ and $\vec{c}_i^n(k)$.
- Repeat until convergence.

Adjustments (A)

- $A^n(k) =$

$$\begin{bmatrix} a_{0,0} + \frac{\xi \Delta t}{2} & 0 & 0 & \cdots & 0 \\ a_{-1,1} & a_{0,1} + \frac{\xi \Delta t}{2} & a_{1,1} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{-1,M-1} & a_{0,M-1} + \frac{\xi \Delta t}{2} & a_{1,M-1} \\ 0 & \cdots & 0 & 0 & a_{0,M} + \frac{\xi \Delta t}{2} \end{bmatrix}$$

Adjustments (c)

- $$\vec{c}_0^{\,n}(k) = \begin{bmatrix} 1 + \xi\Delta t\left(\frac{V_{1,0}^{n+1}(k-1)}{2} + \frac{V_{1,0}^{n} - V_{0,0}^{n}}{2} - k_b\right) \\ c_{0,1}^{n} + \xi\Delta t\left(\frac{V_{1,1}^{n+1}(k-1)}{2} + \frac{V_{1,1}^{n} - V_{0,1}^{n}}{2} - k_b\right) \\ \vdots \\ c_{0,M-1}^{n} + \xi\Delta t\left(\frac{V_{1,M-1}^{n+1}(k-1)}{2} + \frac{V_{1,M-1}^{n} - V_{0,M-1}^{n}}{2} - k_b\right) \\ 1 + \xi\Delta t\left(\frac{V_{1,M}^{n+1}(k-1)}{2} + \frac{V_{1,M}^{n} - V_{0,M}^{n}}{2} - k_b\right) \end{bmatrix}$$
- $$\vec{c}_1^{\,n}(k) = \begin{bmatrix} 1 + \xi\Delta t\left(\frac{V_{0,0}^{n+1}(k-1)}{2} + \frac{V_{0,0}^{n} - V_{1,0}^{n}}{2} + k_s\right) \\ c_{1,1}^{n} + \xi\Delta t\left(\frac{V_{0,1}^{n+1}(k-1)}{2} + \frac{V_{0,1}^{n} - V_{1,1}^{n}}{2} + k_s\right) \\ \vdots \\ c_{1,M-1}^{n} + \xi\Delta t\left(\frac{V_{0,M-1}^{n+1}(k-1)}{2} + \frac{V_{0,M-1}^{n} - V_{1,M-1}^{n}}{2} + k_s\right) \\ 1 + \xi\Delta t\left(\frac{V_{0,M}^{n+1}(k-1)}{2} + \frac{V_{0,M}^{n} - V_{1,M}^{n}}{2} + k_s\right) \end{bmatrix}$$

Convergence

- $\left| \frac{\left\| V_i^{n+1}(k) \right\|}{\left\| V_i^{n+1}(k-1) \right\|} - 1 \right| < \varepsilon$

Miscellaenous

Model Estimation

- Estimate the transition probabilities in the HMM by EM.
- Estimate μ and σ.
  - The log of the prices are Gaussian.
  - $\hat{\sigma}^2$: sample variance
    - $\hat{\sigma}^2 = \sigma^2 \Delta t$
    - $\sigma = \hat{\sigma}\sqrt{\frac{1}{\Delta t}}$
  - $\widehat{m}_i$: sample mean
    - $\widehat{m}_i = \left(\mu_i - \frac{\sigma^2}{2}\right)\Delta t$
    - $\mu_i = \frac{1}{\Delta t}\widehat{m}_i + \frac{\sigma^2}{2}$