

IME

NUMERICAL METHOD

Introduction to Algorithmic Trading Strategies Lecture 3

Trend Following

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References

- ▶ Introduction to Stochastic Calculus with Applications. Fima C Klebaner. 2nd Edition.
- ▶ Estimating continuous time transition matrices from discretely observed data. Inamura, Yasunari. April 2006.
- ▶ Optimal Trend Following Trading Rules. Dai, Min and Zhang, Qing and Zhu, Qiji Jim. 2011.

Stochastic Calculus

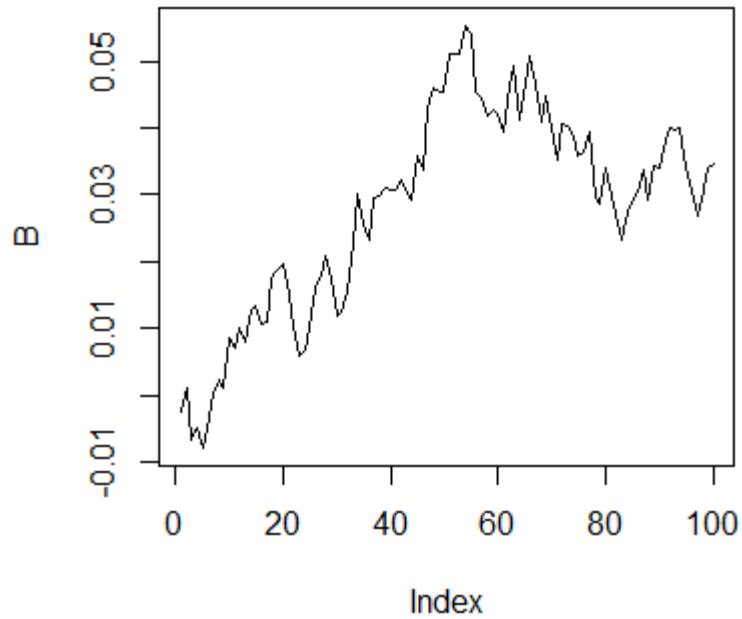
Brownian Motion

- ▶ Independence of increments.
 - ▶ $dB = B(t) - B(s)$ is independent of any history up to s .
- ▶ Normality of increments.
 - ▶ dB is normally distributed with mean 0 and variance $t - s$.
- ▶ Continuity of paths.
 - ▶ $B(t)$ is a continuous function of t .

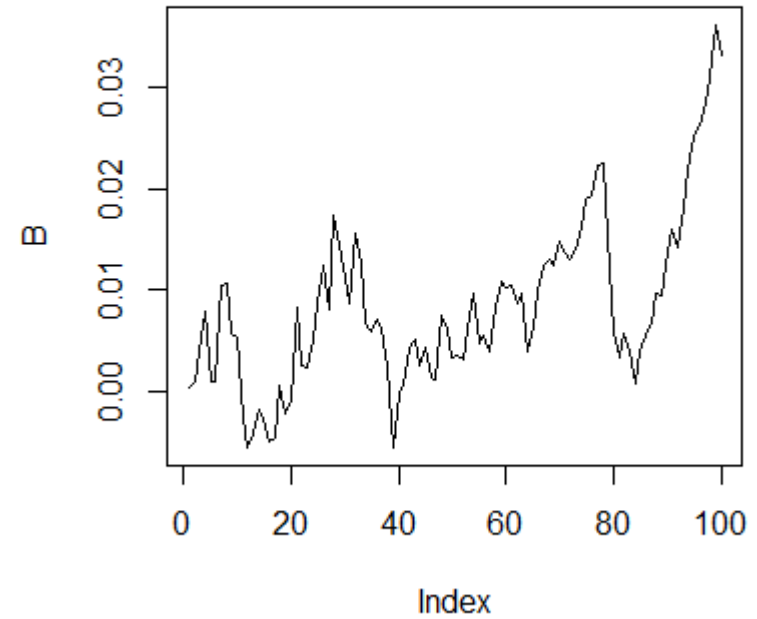
Stochastic vs. Newtonian Calculus

- ▶ Newtonian calculus: when you zoom in a function enough, the little segment looks like a straight line, hence the approximation:
 - ▶ $dy = \dot{f} dt$
 - ▶ Derivative exists.
 - ▶ The function is differentiable.
- ▶ Stochastic calculus: no matter how much you zoom in or how small dt is, the function still looks very zig-zag and random. It is nothing like a straight line.
 - ▶ For Brownian motion, however much you zoom in, how small the segment is, it still just looks very much like a Brownian motion.
 - ▶ The function is therefore no where differentiable.

Examples



not a trend



not mean reversion

dBdB

- ▶ $(dB)^2 = dBdB = dt$
- ▶ $\int_0^t (dB_t)^2 \approx \sum_{i=1}^n Z_{n,i}^2$
 - ▶ $Z_{n,i}$ is of $N\left(0, \frac{t}{n}\right)$ for all i .
- ▶ $\int_0^t (dB_t)^2 \approx$ sum of variances of $Z_{n,i} \approx t$
- ▶ Making $n \rightarrow \infty$ or $dt \rightarrow 0$, we have
 - ▶ $\int_0^t (dB_t)^2 = t$, convergence in probability
 - ▶ $(dB_t)^2 = dt$, in differential form
- ▶ $dBdt = 0$
- ▶ $dt dt = 0$

Asset Price Model

- ▶ Want to model asset price movement.
- ▶ Change in price = dS
 - ▶ Change in price is not too meaningful as \$1 change in a penny stock is more significant than \$1 change in GOOG.
- ▶ Return = $\frac{dS}{S}$
- ▶ Model return using two parts.
 - ▶ Deterministic: μdt , the predictable part. E.g., fixed deposit interest rate.
 - ▶ Random/stochastic: σdB , where σ is the volatility of returns and dB is a sample from a probability distribution, e.g., Normal.

Geometric Brownian Motion

- ▶ Asset price: $\frac{dS}{S} = \mu dt + \sigma dB$
 - ▶ dB : normally distributed
 - ▶ $E(dB) = 0$
 - ▶ Variance = dt . It is intuitive that dB should be scaled by dt otherwise the (random) return drawn would be too big from any Normal distribution for $dt \rightarrow 0$.
 - ▶ $dB = \phi\sqrt{dt}$, where ϕ is a standard Normal distribution.
- ▶ Reasonably good model for stocks and indices.
 - ▶ Real data have more big rises and falls than this model predicts, i.e., extreme events.

GBM Properties

- ▶ Markov property: the distribution of the next price $S + dS$ depend only on the current price S .
- ▶ $E(dS) = E(\mu S dt + \sigma S dB)$
 - ▶ $= E(\mu S dt) + E(\sigma S dB)$
 - ▶ $= \mu S dt + \sigma S E(dB)$
 - ▶ $= \mu S dt$
- ▶ $\text{Var}(dS) = E(dS^2) - E(dS)^2 = \sigma^2 S^2 dt$
 - ▶ $dB dt = 0$
 - ▶ $dt dt = 0$
 - ▶ $dB dB = dt$

Stochastic Differential Equation

- ▶ Both μ and σ can be as simple as constants, deterministic functions of t and S , or as complicated as stochastic functions adapted to the filtration generated by $\{S_t\}$.
- ▶ $dS_t = \mu dt + \sigma dB_t$
- ▶ $dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dB_t$

Univariate Ito's Lemma

- ▶ Assume

- ▶ $dX_t = \mu_t dt + \sigma_t dB_t$

- ▶ $f(t, X_t)$ is twice differentiable of two real variables

- ▶ We have

- ▶
$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma_t \frac{\partial f}{\partial x} dB_t$$

“Proof”

▶ Taylor series

$$\text{▶ } df(t, X)$$

$$\text{▶ } = f_t dt + f_X dX + \frac{1}{2}(f_{tt} dt dt + f_{tX} dt dX + f_{Xt} dX dt + f_{XX} dX dX)$$

$$\text{▶ } = f_t dt + f_X dX + \frac{1}{2} f_{XX} dX dX$$

$$\text{▶ } = f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX}(\mu dt + \sigma dB)^2$$

$$\text{▶ } = f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX}(\mu dt + \sigma dB)^2$$

$$\text{▶ } = f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX}(\mu^2 dt^2 + \sigma^2 dB^2 + 2\mu dt \sigma dB)$$

$$\text{▶ } = f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX} \sigma^2 dB^2$$

$$\text{▶ } = f_t dt + f_X(\mu dt + \sigma dB) + \frac{1}{2} f_{XX} \sigma^2 dt$$

$$\text{▶ } = \left(f_t + \mu f_X + \frac{1}{2} \sigma^2 f_{XX} \right) dt + \sigma f_X dB$$

Log Example

- ▶ For G.B.M., $dX_t = \mu X_t dt + \sigma X_t dz_t$, $d \log X_t = ?$
- ▶ $f(x) = \log(x)$
- ▶ $\frac{\partial f}{\partial t} = 0$
- ▶ $\frac{\partial f}{\partial x} = \frac{1}{x}$
- ▶ $\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$
- ▶ $d \log X_t = \left(\mu X_t \frac{1}{X_t} + \frac{(\sigma X_t)^2}{2} \left(-\frac{1}{X_t^2} \right) \right) dt + \sigma X_t \left(\frac{1}{X_t} \right) dB_t$
- ▶ $= \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dB_t$

Exp Example

▶ $dX_t = \mu dt + \sigma dz_t, de^{X_t} = ?$

▶ $f(x) = e^x$

▶ $\frac{\partial f}{\partial t} = 0$

▶ $\frac{\partial f}{\partial x} = e^x$

▶ $\frac{\partial^2 f}{\partial x^2} = e^x$

▶ $de^{X_t} = \left(\mu e^{X_t} + \frac{\sigma^2}{2} e^{X_t} \right) dt + \sigma e^{X_t} dB_t$

▶ $= e^{X_t} \left[\left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dB_t \right]$

▶ $\frac{de^{X_t}}{e^{X_t}} = \left(\mu + \frac{\sigma^2}{2} \right) dt + \sigma dB_t$

Stopping Time

- ▶ A stopping time is a random time that some event is known to have occurred.
- ▶ Examples:
 - ▶ when a stock hits a target price
 - ▶ when you run out of money
 - ▶ when a moving average crossover occurs
- ▶ When a stock has bottomed out is NOT a stopping time because you cannot tell when it has reached its minimum without future information.

Optimal Trend Following Strategy

Asset Model

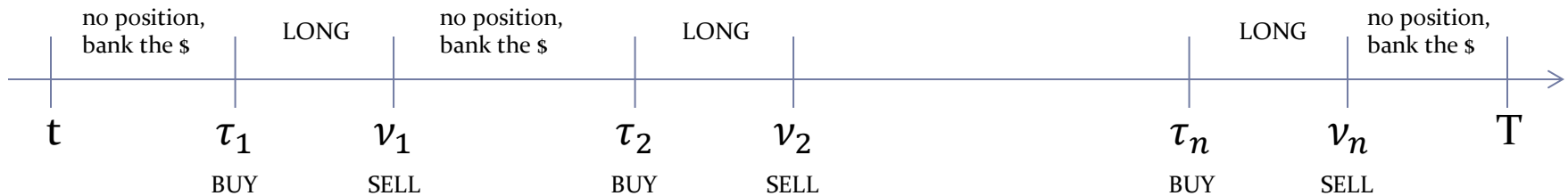
- ▶ Two state Markov model for a stock's prices: BULL and BEAR.
- ▶ $dS_r = S_r[\mu_{\alpha_r} dr + \sigma dB_r]$, $t \leq r \leq T < \infty$
 - ▶ The trading period is between time $[t, T]$.
- ▶ $\alpha_r = \{1, 2\}$ are the two Markov states that indicates the BULL and BEAR markets.
 - ▶ $\mu_1 > 0$
 - ▶ $\mu_2 < 0$
- ▶ $Q = \begin{bmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{bmatrix}$, the generator matrix for the Markov chain.

Generator Matrix

- ▶ From transition matrix to generator matrix:
 - ▶ $P(t, t + \Delta t) = I + \Delta t Q + o(\Delta t)$
- ▶ We can estimate Q by estimating P .
 - ▶ $P(t, s) = P(t, t + m\Delta t)$
 - ▶ $\approx (I + \Delta t Q)^m = \left(I + \frac{(s-t)}{m} Q\right)^m = \exp((s - t)Q)$
 - ▶ $P(t, t + \Delta t) = \exp(\Delta t Q)$
- ▶ $\hat{P}(t) \approx \exp(\Delta t \hat{Q})$
- ▶ $\hat{Q} \approx \frac{1}{\Delta t} \log \hat{P}(t)$
- ▶ E.g., $\Delta t = \frac{1}{252}$

Optimal Stopping Times

$$i = 0, \Lambda_0 = \{\tau_1, \nu_1, \tau_2, \nu_2, \dots\}$$



$$i = 0, \Lambda_0 = \{\nu_1, \tau_2, \nu_2, \tau_3, \dots\}$$

Parameters

- ▶ $\rho \geq 0$, interest free rate
- ▶ $S_t = S$, the initial stock price
- ▶ $i = \{0,1\}$, the net position
- ▶ K_b , the transaction cost for BUYs in percentage
- ▶ K_s , the transaction cost for SELs in percentage

Expected Return (starting flat)

▶ When $i = 0$,

▶ $E_{0,t}(R_t) =$

$$E_t \left(e^{\rho(\tau_1 - t)} \prod_{n=1}^{\infty} \frac{S_{v_n}}{S_{\tau_n}} \left[\frac{1 - K_s}{1 + K_b} \right]^{I_{\{\tau_n < T\}}} e^{\rho(\tau_{n+1} - v_n)} \right)$$

▶ You are long between τ_n and v_n and the return is determined by the price change discounted by the commissions.

▶ You are flat between v_n and τ_{n+1} and the money grows at the risk free rate.

Expected Return (starting long)

▶ When $i = 1$,

▶ $E_{1,t}(R_t) =$

$$E_t \left(\left[\frac{S_{v_1}}{S} e^{\rho(\tau_2 - v_1)} (1 - K_S) \right] \prod_{n=2}^{\infty} \frac{S_{v_n}}{S_{\tau_n}} \left[\frac{1 - K_S}{1 + K_b} \right]^{I_{\{\tau_n < T\}}} e^{\rho(\tau_{n+1} - v_n)} \right)$$

▶ You sell the long position at v_1 .

Value Functions

- ▶ It is easier to work with the log of the returns.

- ▶
$$J_0(S, \alpha, t, \Lambda_0) = E_t \left(\rho(\tau_1 - t) + \sum_{n=1}^{\infty} \left\{ \log \frac{S_{v_n}}{S_{\tau_n}} + I_{\{\tau_n < T\}} \log \frac{1-K_S}{1+K_b} + \rho(\tau_{n+1} - v_n) \right\} \right)$$

- ▶
$$J_1(S, \alpha, t, \Lambda_1) = E_t \left(\left[\log \frac{S_{v_1}}{S} + \rho(\tau_2 - v_1) + \log(1 - K_S) \right] + \sum_{n=2}^{\infty} \left\{ \log \frac{S_{v_n}}{S_{\tau_n}} + I_{\{\tau_n < T\}} \log \frac{1-K_S}{1+K_b} + \rho(\tau_{n+1} - v_n) \right\} \right)$$

Conditional Probability of Bull Market

- ▶ $p_r = P(\alpha_r = 1 \mid \sigma\{S_u : 0 \leq u \leq r\})$
- ▶ p_r is therefore a random process driven by the same Brownian motion that drives S_u .
- ▶ $dp_r = [-(\lambda_1 + \lambda_2)p_r + \lambda_2]dr + \frac{(\mu_1 - \mu_2)p_r(1 - p_r)}{\sigma} d\widehat{B}_r$
 - ▶ $d\widehat{B}_r = \frac{d \log(S_r) - [(\mu_1 - \mu_2)p_r + \mu_2 - \sigma^2/2]dr}{\sigma}$
- ▶ $p_{t+1} = p_t + g(p_t)\Delta t + \frac{(\mu_1 - \mu_2)p_t(1 - p_t)}{\sigma^2} \log\left(\frac{S_{t+1}}{S_t}\right)$
 - ▶ $g(p) = -(\lambda_1 + \lambda_2)p + \lambda_2 - \frac{(\mu_1 - \mu_2)p(1 - p)[(\mu_1 - \mu_2)p + \mu_2 - \sigma^2/2]}{\sigma^2}$
 - ▶ Make p_{t+1} between 0 and 1 if it falls outside the bounds.

Objective

- ▶ Find an optimal trading sequence (the stopping times) so that the value functions are maximized.
- ▶ $V_i(p, t) = \sup_{\Lambda_i} J_i(S, p, t, \Lambda_i)$
 - ▶ V_i : the maximum amount of expected returns

Realistic Expectation of Returns

- ▶ Lower bounds:

- ▶ $V_0(p, t) \geq \rho(T - t)$

- ▶ $V_1(p, t) \geq \rho(T - t) + \log(1 - K_S)$

- ▶ Upper bound:

- ▶ $V_i(p, t) \leq \left(\mu_1 - \frac{\sigma^2}{2} \right) (T - t)$

- ▶ No trading zones:

- ▶ One should never buy when $\rho \geq \mu_1 - \frac{\sigma^2}{2}$

Principal of Optimality

- ▶ Given an optimal trading sequence, Λ_i , starting from time t , the truncated sequence will also be optimal for the same trading problem starting from any stopping times τ_n or ν_n .

Coupled Value Functions

$$\begin{cases} V_0(p, t) = \sup_{\tau_1} E_t \{ \rho(\tau_1 - t) - \log(1 + K_b) + V_1(p_{\tau_1}, \tau_1) \} \\ V_1(p, t) = \sup_{v_1} E_t \left\{ \log \frac{S_{v_1}}{S_t} + \log(1 - K_s) + V_0(p_{v_1}, v_1) \right\} \end{cases}$$

Hamilton-Jacobi-Bellman Equations

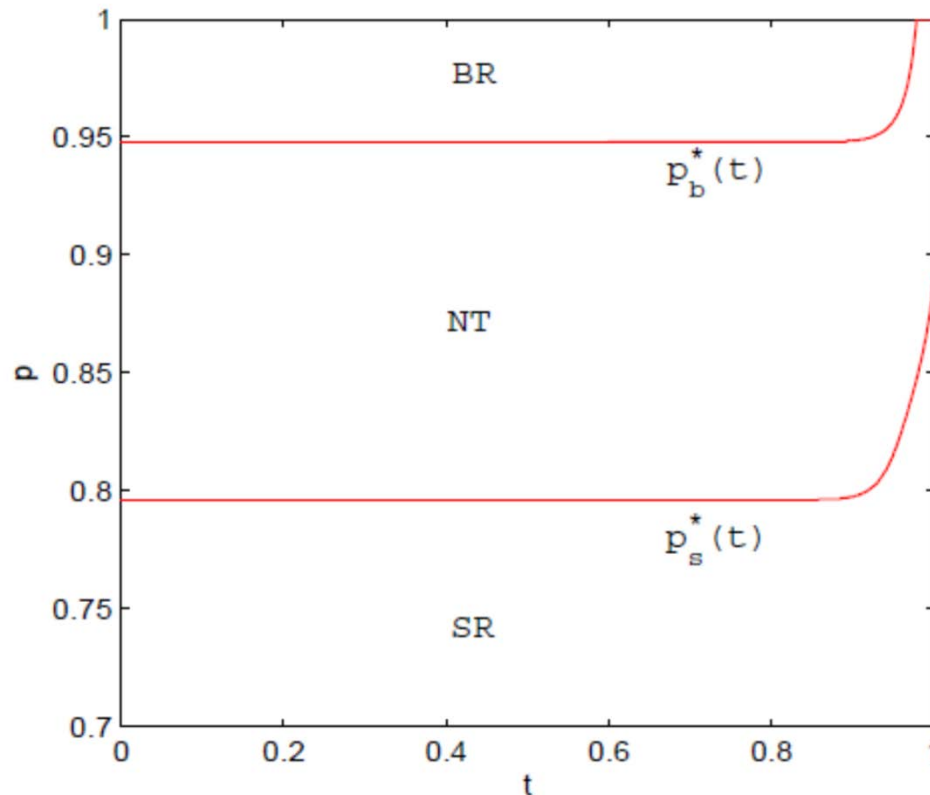
- ▶
$$\begin{cases} \min\{-\mathcal{L}V_0 - \rho, V_0 - V_1 + \log(1 + K_b)\} = 0 \\ \min\{-\mathcal{L}V_1 - f(\rho), V_1 - V_0 - \log(1 - K_s)\} = 0 \end{cases}$$
- ▶ with terminal conditions:
$$\begin{cases} V_0(p, T) = 0 \\ V_1(p, T) = \log(1 - K_s) \end{cases}$$
- ▶
$$\mathcal{L} = \partial_t + \frac{1}{2} \left(\frac{(\mu_1 - \mu_2)p(1-p)}{\sigma} \right)^2 \partial_{pp} + [-(\lambda_1 + \lambda_2)p + \lambda_2] \partial_p$$

Penalty Formulation Approach

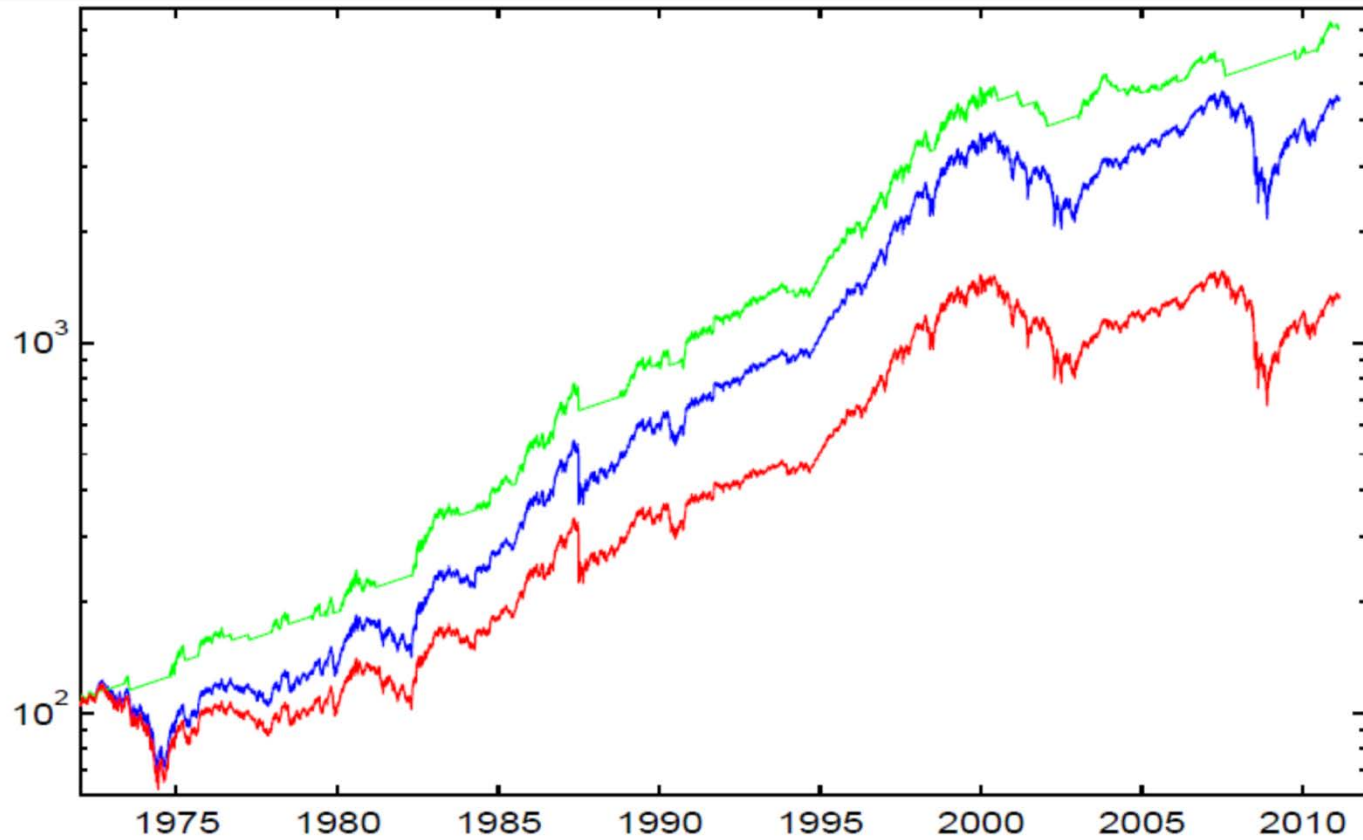
- ▶ The HJB formulation is equivalent to this system of PDEs.
- ▶
$$\begin{cases} -\mathcal{L}V_0 - h(\rho) = \hat{\xi}(V_1 - V_0 - \log(1 + K_b))^+ \\ -\mathcal{L}V_1 - f(\rho) = \hat{\xi}(V_0 - V_1 + \log(1 - K_s))^+ \end{cases}$$
- ▶ $\hat{\xi}$ is a penalization factor such that $\hat{\xi} \rightarrow \infty$.
- ▶ $h(\rho) = \rho$

Trading Boundaries

- ▶ $BR = \{p \in (0,1) \times [0,T) : p \geq p_b^*(t)\}$
- ▶ $SR = \{p \in (0,1) \times [0,T) : p \leq p_s^*(t)\}$

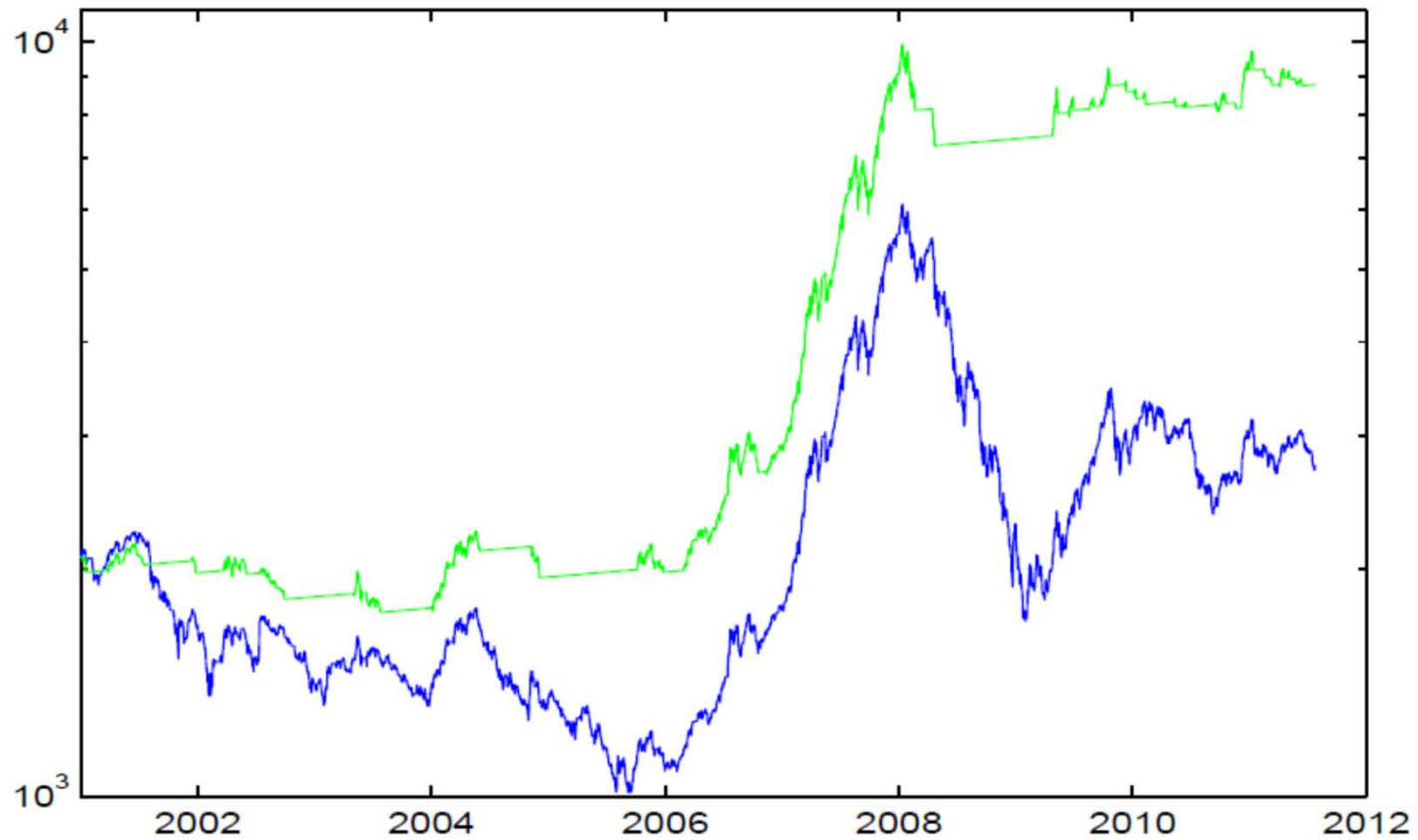


SP500



Trend following trading of SP500 1972–2011 compared with buy and hold

SSE



Trend following trading of SSE 2001–2011 compared with buy and hold

Problems

- ▶ The buy entry signals are always delayed (as expected).
- ▶ The market therefore needs to bull long enough for the strategy to make profit.
- ▶ If the bear market comes in too fast and hard, the exit entry may not come in soon enough.
 - ▶ A proper stoploss from max drawdown may help.

Finding the Boundaries

- ▶ At any time t , we need to find p_b^* and p_s^* to determine whether we buy or sell or go flat.
- ▶ p_b^* is determined by
 - ▶ $V_1(p_b^*, t) - V_0(p_b^*, t) = \log(1 + K_b)$
- ▶ p_s^* is determined by
 - ▶ $V_1(p_s^*, t) - V_0(p_s^*, t) = \log(1 - K_s)$
- ▶ Need to compute $V_0(p, t)$ and $V_1(p, t)$ for all p and t .
 - ▶ Solve the coupled PDEs.

Discretization of \mathcal{L}

▶ $(\rho, t) \in [0,1] \times [0,1)$

▶ (ρ, t) as $\{(\rho_j, t_n)\}$

▶ $j \in \{0, \dots, M\}$

▶ $\rho_0 = 0$

▶ $\rho_M = 1$

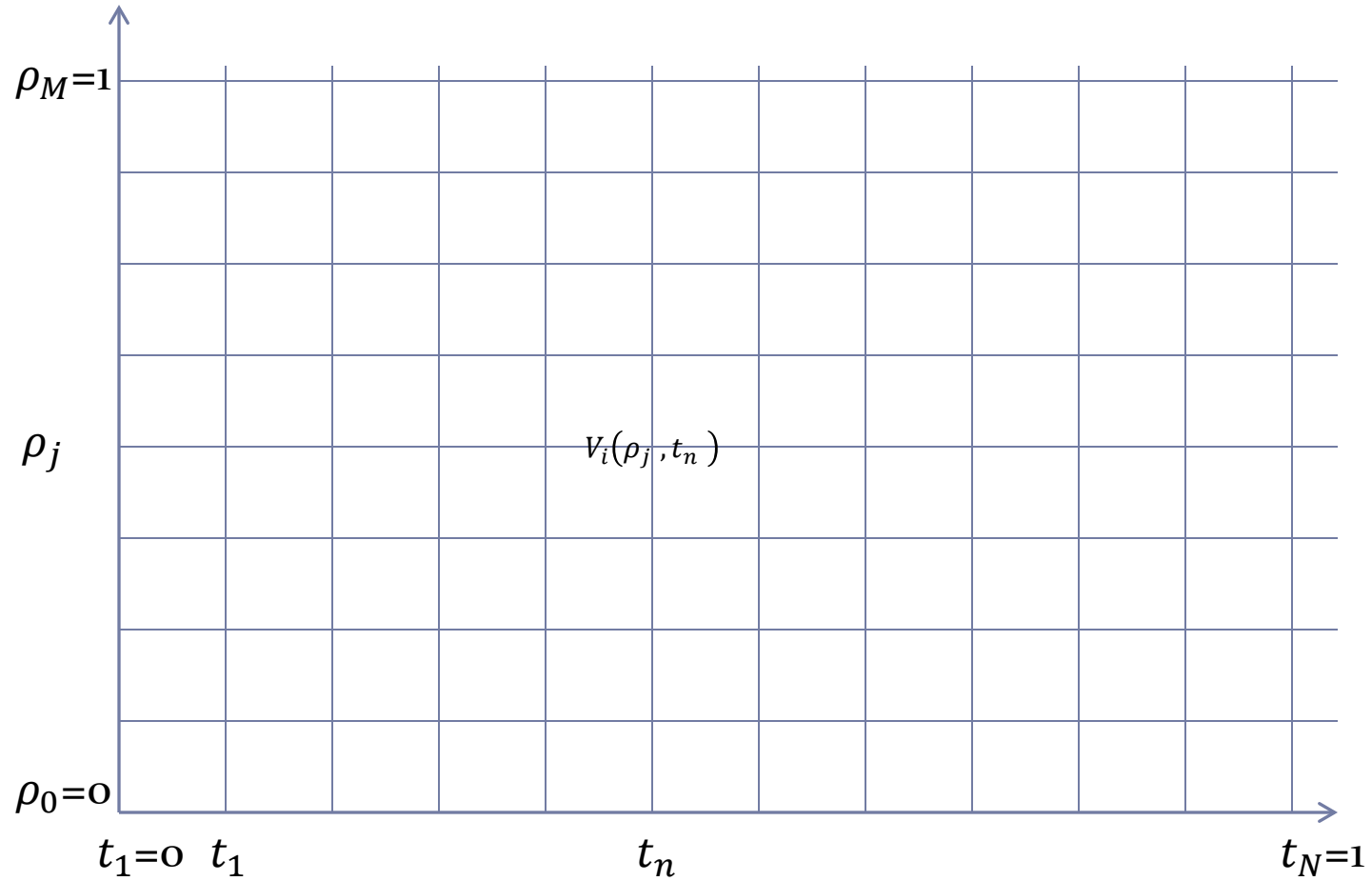
▶ $n \in \{1, \dots, N\}$

▶ $t_1 = 0$

▶ $t_N = 1^-$

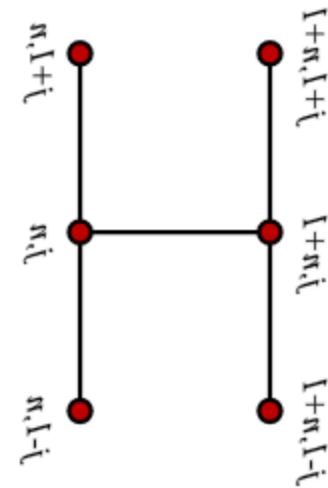
▶ $\frac{\partial V_i}{\partial t} \approx \frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t}$

The Grid



Crank-Nicolson Scheme

$$\triangleright \frac{\partial V_i}{\partial \rho} \approx \frac{1}{2} \left(\frac{V_{i,j+1}^{n+1} - V_{i,j-1}^{n+1}}{2\Delta\rho} + \frac{V_{i,j+1}^n - V_{i,j-1}^n}{2\Delta\rho} \right)$$



$$\triangleright \frac{\partial^2 V_i}{\partial \rho^2} \approx \frac{1}{2} \left(\frac{V_{i,j+1}^{n+1} - 2V_{i,j}^{n+1} + V_{i,j-1}^{n+1}}{(\Delta\rho)^2} + \frac{V_{i,j+1}^n - 2V_{i,j}^n + V_{i,j-1}^n}{(\Delta\rho)^2} \right)$$

RHS

$$\triangleright \hat{\xi}(V_1 - V_0 - \log(1 + K_b))^+ = \xi_{0,j}^{n+1/2} \left(\frac{V_{1,j}^{n+1} - V_{0,j}^{n+1}}{2} + \frac{V_{1,j}^n - V_{0,j}^n}{2} - k_b \right)$$

$$\triangleright k_b = \log(1 + K_b)$$

$$\triangleright \hat{\xi}(V_0 - V_1 + \log(1 - K_s))^+ = \xi_{1,j}^{n+1/2} \left(\frac{V_{0,j}^{n+1} - V_{1,j}^{n+1}}{2} + \frac{V_{0,j}^n - V_{1,j}^n}{2} + k_s \right)$$

$$\triangleright k_s = \log(1 - K_s)$$

Penalty

$$\blacktriangleright \hat{\xi}_{0,j}^{n+1/2} = \begin{cases} \xi \Delta t, & \text{if } \frac{V_{1,j}^{n+1} - V_{0,j}^{n+1}}{2} + \frac{V_{1,j}^n - V_{0,j}^n}{2} - k_b > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\blacktriangleright \hat{\xi}_{1,j}^{n+1/2} = \begin{cases} \xi \Delta t, & \text{if } \frac{V_{0,j}^{n+1} - V_{1,j}^{n+1}}{2} + \frac{V_{0,j}^n - V_{1,j}^n}{2} + k_s > 0 \\ 0, & \text{otherwise} \end{cases}$$

The Discretized PDEs

- ▶ $a_{1,j}V_{0,j+1}^{n+1} + a_{0,j}V_{0,j}^{n+1} + a_{-1,j}V_{0,j-1}^{n+1} =$
 $c_{0,j}^n + \xi_{0,j}^{n+1/2} \left(\frac{V_{1,j}^{n+1} - V_{0,j}^{n+1}}{2} + \frac{V_{1,j}^n - V_{0,j}^n}{2} - k_b \right)$
- ▶ $a_{1,j}V_{1,j+1}^{n+1} + a_{0,j}V_{1,j}^{n+1} + a_{-1,j}V_{1,j-1}^{n+1} =$
 $c_{1,j}^n + \xi_{1,j}^{n+1/2} \left(\frac{V_{0,j}^{n+1} - V_{1,j}^{n+1}}{2} + \frac{V_{0,j}^n - V_{1,j}^n}{2} + k_s \right)$
- ▶ $j \in \{1, \dots, M - 1\}$
 - ▶ $p = 0, p = 1$ are excluded.

The Coefficients

- ▶ $a_{-1,j} = -\frac{1}{2}\sigma_p^2 \frac{\Delta t}{2\Delta p^2} + \frac{1}{2}\mu_p \frac{\Delta t}{2\Delta p}$
- ▶ $a_{0,j} = 1 + \sigma_p^2 \frac{\Delta t}{2\Delta p^2}$
- ▶ $a_{1,j} = -\frac{1}{2}\sigma_p^2 \frac{\Delta t}{2\Delta p^2} - \frac{1}{2}\mu_p \frac{\Delta t}{2\Delta p}$
- ▶ $c_{0,j}^n = -a_{1,j}V_{0,j+1}^n + (2 - a_{0,j})V_{0,j}^n - a_{-1,j}V_{0,j-1}^n + h(j\Delta p)\Delta t$
- ▶ $c_{1,j}^n = -a_{1,j}V_{1,j+1}^n + (2 - a_{0,j})V_{1,j}^n - a_{-1,j}V_{1,j-1}^n + f(j\Delta p)\Delta t$

Boundary Conditions

- ▶ $j = 0, p = 0$
 - ▶ $a_{0,0} = 1$
 - ▶ $a_{1,0} = 0$
 - ▶ $c_{0,0}^n = V_{0,1}^n + h(0)\Delta t$
 - ▶ $c_{1,0}^n = V_{1,1}^n + f(0)\Delta t$
- ▶ $j = M, p = 1$
 - ▶ $a_{0,M} = 1$
 - ▶ $a_{-1,M} = 0$
 - ▶ $c_{0,M}^n = V_{0,M-1}^n + h(1)\Delta t$
 - ▶ $c_{1,M}^n = V_{1,M-1}^n + f(1)\Delta t$
- ▶ $n = 0, t = 0, \text{ for all } j \in \{0, \dots, M\}$
 - ▶ $V_{0,j} = 0$
 - ▶ $V_{1,j} = k_s$

System in Matrix Form

$$\begin{cases} A^n \vec{V}_0^{n+1} = \vec{c}_0^n \\ A^n \vec{V}_1^{n+1} = \vec{c}_1^n \end{cases}$$

$$A^n = \begin{bmatrix} a_{0,0} & 0 & 0 & \dots & 0 \\ a_{-1,1} & a_{0,1} & a_{1,1} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & a_{-1,M-1} & a_{0,M-1} & a_{1,M-1} \\ 0 & \dots & 0 & 0 & a_{0,M} \end{bmatrix}$$

Solving the System of Equations

- ▶ We already know the values of \vec{V}_0^0 and \vec{V}_1^0 .
- ▶ Starting from $n = 0$, using A^n , we iteratively solve for \vec{V}_0^{n+1} and \vec{V}_1^{n+1} until $n = M - 1$.
 - ▶ There are totally M systems of equations to solve.
- ▶ The equations are linear in the unknowns $V_{0,j}^{n+1}$ and $V_{1,j}^{n+1}$ if and only if $\hat{\xi}_{0,j}^{n+1/2} = 0$.
- ▶ When the equations are linear, we can use Thomas' algorithm to solve for a tri-diagonal system of linear equations.
- ▶ Otherwise, we use an iterative scheme.

Iterative Scheme

- ▶ 1st step, initialization, $k = 0$
 - ▶ Assume $\hat{\xi}_{i,j}^{n+1/2}(0) = 0$. Solve for $\vec{V}_i^{n+1}(0)$.
- ▶ kth step
 - ▶ Using $\vec{V}_i^{n+1}(k-1)$, update $\hat{\xi}_{i,j}^{n+1/2}(k)$.
 - ▶ When $\hat{\xi}_{i,j}^{n+1/2}(k) > 0$, adjust the $A^n(k)$ and $\vec{c}_i^n(k)$.
- ▶ Repeat until convergence.

Adjustments (A)

▶ $A^n(k) =$

$$\begin{bmatrix} a_{0,0} + \frac{\xi\Delta t}{2} & 0 & 0 & \dots & 0 \\ a_{-1,1} & a_{0,1} + \frac{\xi\Delta t}{2} & a_{1,1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & a_{-1,M-1} & a_{0,M-1} + \frac{\xi\Delta t}{2} & a_{1,M-1} \\ 0 & \dots & 0 & 0 & a_{0,M} + \frac{\xi\Delta t}{2} \end{bmatrix}$$

Adjustments (c)

$$\blacktriangleright \vec{c}_0^n(k) = \begin{bmatrix} 1 + \xi\Delta t \left(\frac{V_{1,0}^{n+1}(k-1)}{2} + \frac{V_{1,0}^n - V_{0,0}^n}{2} - k_b \right) \\ c_{0,1}^n + \xi\Delta t \left(\frac{V_{1,1}^{n+1}(k-1)}{2} + \frac{V_{1,1}^n - V_{0,1}^n}{2} - k_b \right) \\ \vdots \\ c_{0,M-1}^n + \xi\Delta t \left(\frac{V_{1,M-1}^{n+1}(k-1)}{2} + \frac{V_{1,M-1}^n - V_{0,M-1}^n}{2} - k_b \right) \\ 1 + \xi\Delta t \left(\frac{V_{1,M}^{n+1}(k-1)}{2} + \frac{V_{1,M}^n - V_{0,M}^n}{2} - k_b \right) \end{bmatrix}$$

$$\blacktriangleright \vec{c}_1^n(k) = \begin{bmatrix} 1 + \xi\Delta t \left(\frac{V_{0,0}^{n+1}(k-1)}{2} + \frac{V_{0,0}^n - V_{1,0}^n}{2} + k_s \right) \\ c_{1,1}^n + \xi\Delta t \left(\frac{V_{0,1}^{n+1}(k-1)}{2} + \frac{V_{0,1}^n - V_{1,1}^n}{2} + k_s \right) \\ \vdots \\ c_{1,M-1}^n + \xi\Delta t \left(\frac{V_{0,M-1}^{n+1}(k-1)}{2} + \frac{V_{0,M-1}^n - V_{1,M-1}^n}{2} + k_s \right) \\ 1 + \xi\Delta t \left(\frac{V_{0,M}^{n+1}(k-1)}{2} + \frac{V_{0,M}^n - V_{1,M}^n}{2} + k_s \right) \end{bmatrix}$$

Convergence

$$\triangleright \left| \frac{\|V_i^{n+1}(k)\|}{\|V_i^{n+1}(k-1)\|} - 1 \right| < \varepsilon$$



Miscellaenous

Model Estimation

- ▶ Estimate the transition probabilities in the HMM by EM.
- ▶ Estimate μ and σ .
 - ▶ The log of the prices are Gaussian.
 - ▶ $\hat{\sigma}^2$: sample variance
 - ▶ $\hat{\sigma}^2 = \sigma^2 \Delta t$
 - ▶ $\sigma = \hat{\sigma} \sqrt{\frac{1}{\Delta t}}$
 - ▶ \hat{m}_i : sample mean
 - ▶ $\hat{m}_i = \left(\mu_i - \frac{\sigma^2}{2} \right) \Delta t$
 - ▶ $\mu_i = \frac{1}{\Delta t} \hat{m}_i + \frac{\sigma^2}{2}$