Introduction to Algorithmic Trading Strategies
Lecture 4

Optimal Pairs Trading by Stochastic Control

Haksun Li
haksun.li@numericalmethod.com
www.numericalmethod.com

Outline

- Problem formulation
- Ito's lemma
- Dynamic programming
- Hamilton-Jacobi-Bellman equation
- Riccati equation
- Integrating factor

Reference

- Optimal Pairs Trading: A Stochastic Control Approach. Mudchanatongsuk, S., Primbs, J.A., Wong, W. Dept. of Manage. Sci. & Eng., Stanford Univ., Stanford, CA.

Basket Creation vs. Trading

- In lecture 3, we discussed a few ways to construct a mean-reverting basket.
- In this and the next lectures, we discuss how to trade a mean-reverting asset, if such exists.

Stochastic Control

- We model the difference between the log-returns of two assets as an Ornstein-Uhlenbeck process.
- We compute the optimal position to take as a function of the deviation from the equilibrium.
- This is done by solving the corresponding the Hamilton-Jacobi-Bellman equation.

Formulation

- Assume a risk free asset M_t, which satisfies
  - $dM_t = rM_t dt$
- Assume two assets, A_t and B_t.
- Assume B_t follows a geometric Brownian motion.
  - $dB_t = \mu B_t dt + \sigma B_t dz_t$
- x_t is the spread between the two assets.
  - $x_t = \log A_t - \log B_t$

Ornstein-Uhlenbeck Process

- We assume the spread, the basket that we want to trade, follows a mean-reverting process.
  - $dx_t = k(\theta - x_t)dt + \eta d\omega_t$
- θ is the long term equilibrium to which the spread reverts.
- k is the rate of reversion. It must be positive to ensure stability around the equilibrium value.

Instantaneous Correlation

- Let ρ denote the instantaneous correlation coefficient between z and ω.
  - $E[d\omega_t dz_t] = \rho dt$

Univariate Ito's Lemma

- Assume
  - $dX_t = \mu_t dt + \sigma_t dB_t$
  - $f(t, X_t)$ is twice differentiable of two real variables
- We have
  - $df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{{\sigma_t}^2}{2} \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma_t \frac{\partial f}{\partial x} dB_t$

Log example

- For G.B.M., $dX_t = \mu X_t dt + \sigma X_t dz_t$, $d\log X_t = ?$
- $f(x) = \log(x)$
- $\frac{\partial f}{\partial t} = 0$
- $\frac{\partial f}{\partial x} = \frac{1}{x}$
- $\frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$
- $d\log X_t = \left(\mu X_t \frac{1}{X_t} + \frac{(\sigma X_t)^2}{2}\left(-\frac{1}{{X_t}^2}\right)\right) dt + \sigma X_t \left(\frac{1}{X_t}\right) dB_t$
- $= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB_t$

Multivariate Ito's Lemma

- Assume
  - $X_t = (X_{1t}, X_{2t}, \cdots, X_{nt})$ is a vector Ito process
  - $f(x_{1t}, x_{2t}, \cdots, x_{nt})$ is twice differentiable
- We have
  - $df(X_{1t}, X_{2t}, \cdots, X_{nt})$
  - $= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(X_{1t}, X_{2t}, \cdots, X_{nt}) dX_i(t)$
  - $+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} f(X_{1t}, X_{2t}, \cdots, X_{nt}) d[X_i, X_j](t)$

Multivariate Example

- $\log A_t = x_t + \log B_t$
- $A_t = \exp(x_t + \log B_t)$
- $\frac{\partial A_t}{\partial x_t} = \exp(x_t + \log B_t) = A_t$
- $\frac{\partial A_t}{\partial B_t} = \exp(x_t + \log B_t)\frac{1}{B_t} = \frac{A_t}{B_t}$
- $\frac{\partial^2 A_t}{\partial {x_t}^2} = \frac{\partial A_t}{\partial x_t} = A_t$
- $\frac{\partial^2 A_t}{\partial {B_t}^2} = \frac{\partial}{\partial B_t}\left(\frac{A_t}{B_t}\right) = 0$
- $\frac{\partial^2 A_t}{\partial B_t \partial x_t} = \frac{\partial}{\partial B_t}\left(\frac{\partial A_t}{\partial x_t}\right) = \frac{\partial A_t}{\partial B_t} = \frac{A_t}{B_t}$

What is the Dynamic of Asset A_t?

- $\partial A_t = \frac{\partial A_t}{\partial x} dx_t + \frac{\partial A_t}{\partial B_t} dB_t + \frac{1}{2}\frac{\partial^2 A_t}{\partial x^2}(dx_t)^2 + \frac{\partial^2 A_t}{\partial B_t \partial x}(dx_t)(dB_t)$
- $= A_t dx_t + \frac{A_t}{B_t} dB_t + \frac{1}{2} A_t (dx_t)^2 + \frac{A_t}{B_t}(dx_t)(dB_t)$
- $= A_t[k(\theta - x_t)dt + \eta d\omega_t] + \frac{A_t}{B_t}[\mu B_t dt + \sigma B_t dz_t] + \frac{1}{2} A_t \eta^2 dt + \frac{A_t}{B_t} \rho\eta\sigma B_t dt$

Dynamic of Asset A_t

- $\partial A_t = A_t[k(\theta - x_t)dt + \eta d\omega_t] + A_t[\mu dt + \sigma dz_t] + \frac{1}{2}A_t\eta^2 dt + A_t\rho\eta\sigma dt$
- $= A_t\left[k(\theta - x_t) + \mu + \frac{1}{2}\eta^2 + \rho\eta\sigma\right]dt + A_t\eta d\omega_t + A_t\sigma dz_t$
- $= A_t\left\{\left[\mu + k(\theta - x_t) + \frac{1}{2}\eta^2 + \rho\eta\sigma\right]dt + \sigma dz_t + \eta d\omega_t\right\}$

Notations

- V_t: the value of a self-financing pairs trading portfolio
- h_t:the portfolio weight for stock A
- $\widetilde{h_t} = -h_t$:the portfolio weight for stock B

Self-Financing Portfolio Dynamic

- $\frac{dV_t}{V_t} = h_t \frac{dA_t}{A_t} + \widetilde{h_t} \frac{dB_t}{B_t} + \frac{dM_t}{M_t}$
- $= h_t \left\{ \left[\mu + k(\theta - x_t) + \frac{1}{2}\eta^2 + \rho\eta\sigma \right] dt + \sigma dz_t + \eta d\omega_t \right\} - h_t\{\mu dt + \sigma dz_t\} + r dt$
- $= h_t \left\{ \left[k(\theta - x_t) + \frac{1}{2}\eta^2 + \rho\eta\sigma \right] dt + \eta d\omega_t \right\} + r dt$
- $= \left\{ h_t \left[k(\theta - x_t) + \frac{1}{2}\eta^2 + \rho\eta\sigma \right] + r \right\} dt + h_t \eta d\omega_t$

Power Utility

- Investor preference:
  - $U(x) = x^{\gamma}$
  - $x \geq 0$
  - $0 < \gamma < 1$

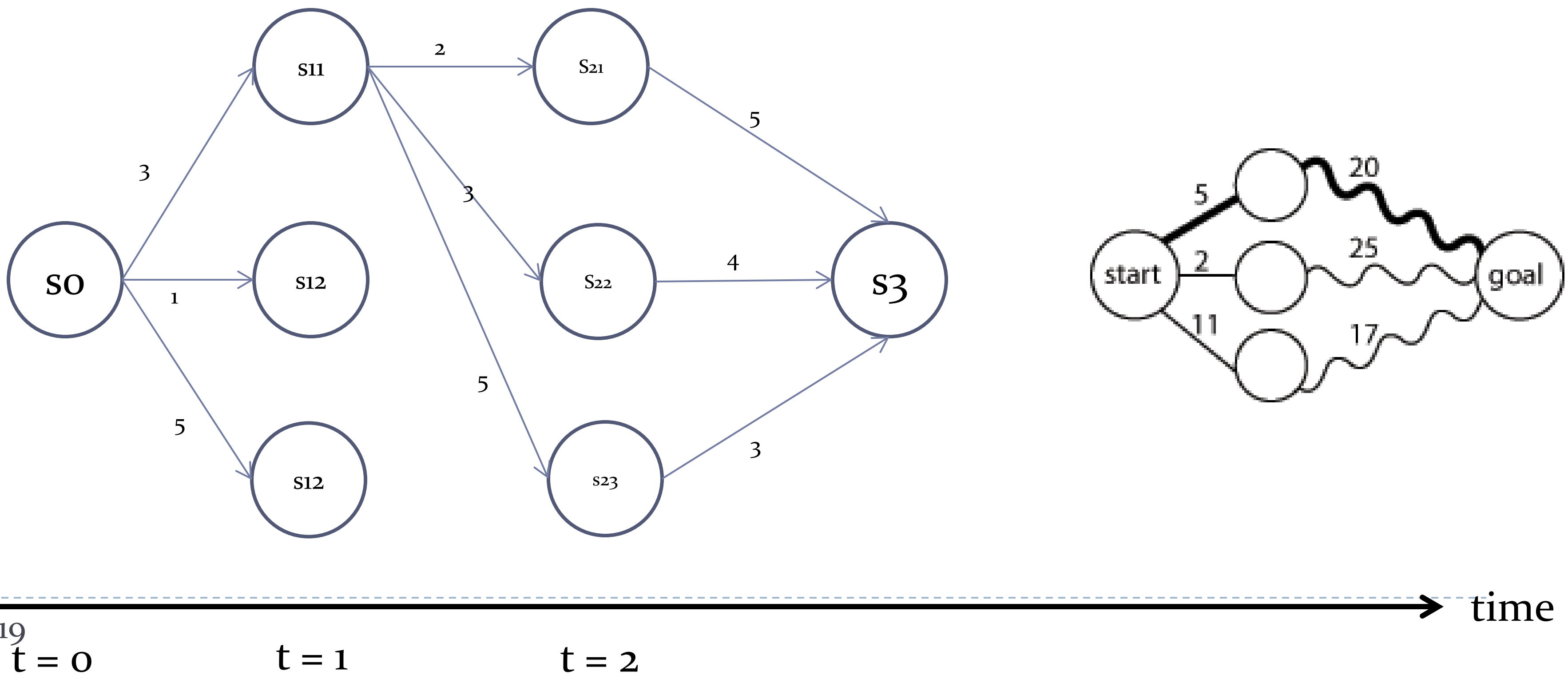
Problem Formulation

- $\max_{h_t} E[{V_T}^{\gamma}]$, s.t.,
- $V(0) = v_0$, $\mathrm{x}(0) = x_0$
  - $dx_t = k(\theta - x_t)dt + \eta d\omega_t$
- $dV_t = h_t dx_t = h_t k(\theta - x_t)dt + h_t \eta d\omega_t$
- Note that we simplify GBM to BM of V_t, and remove some constants.

Dynamic Programming

- Consider a stage problem to minimize (or maximize) the accumulated costs over a system path.

$$\text{Cost} = c_3 + \sum_{t=0}^{2} c_t$$



Dynamic Programming Formulation

- State change: $x_{k+1} = f_k(x_k, u_k, \omega_k)$
  - k: time
  - x_k: state
  - u_k: control decision selected at time k
  - ω_k: a random noise
- Cost: $g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, \omega_k)$
- Objective: minimize (maximize) the expected cost.
  - We need to take expectation to account for the noise, ω_k.

Principle of Optimality

- Let $\pi^* = \{\mu_0{}^*, \mu_1{}^*, \cdots, \mu_{N-1}{}^*\}$ be an optimal policy for the basic problem, and assume that when using π^*, a give state x_i occurs at time i with positive probability. Consider the sub-problem whereby we are at x_i at time i and wish to minimize the "cost-to-go" from time i to time N.
  - $E\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, \omega_k)\right\}$
- Then the truncated policy $\{\mu_i{}^*, \mu_{i+1}{}^*, \cdots, \mu_{N-1}{}^*\}$ is optimal for this sub-problem.

Dynamic Programming Algorithm

- For every initial state x_0, the optimal cost $J^*(x_k)$ of the basic problem is equal to $J_0(x_0)$, given by the last step of the following algorithm, which proceeds backward in time from period $N-1$ to period 0:
  - $J_N(x_N) = g_N(x_N)$
- $J_k(x_k) = \min_{u_k} E\{g_k(x_k, u_k, \omega_k) + J_{k+1}(f_k(x_k, u_k, \omega_k))\}$

Value function

- Terminal condition:
  - $G(T, V, x) = V^{\gamma}$
- DP equation:
  - $G(t, V_t, x_t) = \max_{h_t} E\{G(t + dt, V_{t+dt}, x_{t+dt})\}$
  - $G(t, V_t, x_t) = \max_{h_t} E\{G(t, V_t, x_t) + \Delta G\}$
- By Ito's lemma:
  - $\Delta G = G_t dt + G_V(dV) + G_x(dx) + \frac{1}{2} G_{VV}(dV)^2 + \frac{1}{2} G_{xx}(dx)^2 + G_{Vx}(dV)(dx)$

Hamilton-Jacobi-Bellman Equation

- Cancel $G(t, V_t, x_t)$ on both LHS and RHS.
- Divide by time discretization, Δt.
- Take limit as $\Delta t \to 0$, hence Ito.
- $0 = \max_{h_t} E\{\Delta G\}$
- $\max_{h_t} E\left\{G_t dt + G_V(dV) + G_x(dx) + \frac{1}{2}G_{VV}(dV)^2 + \frac{1}{2}G_{xx}(dx)^2 + G_{Vx}(dV)(dx)\right\} = 0$
- The optimal portfolio position is ${h_t}^*$.

HJB for Our Portfolio Value

- $\max_{h_t} E\left\{G_t dt + G_V(dV) + G_x(dx) + \frac{1}{2}G_{VV}(dV)^2 + \frac{1}{2}G_{xx}(dx)^2 + G_{Vx}(dV)(dx)\right\} = 0$
- $$\max_{h_t} E\left\{\begin{matrix} G_t dt + G_V(h_t k(\theta - x_t)dt + h_t\eta d\omega_t) + G_x(dx) + \\ \frac{1}{2}G_{VV}(h_t k(\theta - x_t)dt + h_t\eta d\omega_t)^2 + \frac{1}{2}G_{xx}(dx)^2 + \\ G_{Vx}(h_t k(\theta - x_t)dt + h_t\eta d\omega_t)(dx) \end{matrix}\right\} = 0$$
- $$\max_{h_t} E\left\{\begin{matrix} G_t dt + G_V(h_t k(\theta - x_t)dt + h_t\eta d\omega_t) + \\ G_x(k(\theta - x_t)dt + \eta d\omega_t) + \\ \frac{1}{2}G_{VV}(h_t k(\theta - x_t)dt + h_t\eta d\omega_t)^2 + \\ \frac{1}{2}G_{xx}(k(\theta - x_t)dt + \eta d\omega_t)^2 + \\ G_{Vx}(h_t k(\theta - x_t)dt + h_t\eta d\omega_t) \times (k(\theta - x_t)dt + \eta d\omega_t) \end{matrix}\right\} = 0$$

Taking Expectation

- All $\eta d\omega_t$ disapper because of the expectation operator.
- Only the deterministic dt terms remain.
- Divide LHR and RHS by dt.
- $$\max_{h_t} \left\{ \begin{matrix} G_t + G_V(h_t k(\theta - x_t)) + \\ G_x\big(k(\theta - x_t)\big) + \frac{1}{2} G_{VV}(h_t \eta)^2 + \frac{1}{2} G_{xx} \eta^2 \\ + G_{Vx}(h_t \eta^2) \end{matrix} \right\} = 0$$

Dynamic Programming Solution

- Solve for the cost-to-go function, G_t.
- Assume that the optimal policy is h_t^*.

First Order Condition

- Differentiate w.r.t. h_t.
- $G_V\left(k(\theta - x_t)\right) + {h_t}^* G_{VV}\eta^2 + G_{Vx}\eta^2 = 0$
- ${h_t}^* = -\frac{G_V\left(k(\theta - x_t)\right) + G_{Vx}\eta^2}{G_{VV}\eta^2}$
- In order to determine the optimal position, ${h_t}^*$, we need to solve for G to get G_V, G_{Vx}, and G_{VV}.

The Partial Differential Equation (1)

$$
\begin{gathered}
G_t - \\
G_V\left(\frac{G_V\big(k(\theta-x_t)\big)+G_{Vx}\eta^2}{G_{VV}\eta^2}k(\theta-x_t)\right) + \\
G_x\big(k(\theta-x_t)\big) + \\
\frac{1}{2}G_{VV}\eta^2\left(\frac{G_V\big(k(\theta-x_t)\big)+G_{Vx}\eta^2}{G_{VV}\eta^2}\right)^2 + \\
\frac{1}{2}G_{xx}\eta^2 - \\
G_{Vx}\eta^2\frac{G_V\big(k(\theta-x_t)\big)+G_{Vx}\eta^2}{G_{VV}\eta^2}
\end{gathered}
= 0
$$

The Partial Differential Equation (2)

$$\begin{aligned} & G_t - \\ & G_V k(\theta - x_t) \frac{G_V\left(k(\theta - x_t)\right) + G_{Vx}\eta^2}{G_{VV}\eta^2} + \\ & G_x\left(k(\theta - x_t)\right) + \\ & \frac{1}{2}\frac{\left[G_V\left(k(\theta - x_t)\right) + G_{Vx}\eta^2\right]^2}{G_{VV}\eta^2} + \\ & \frac{1}{2}G_{xx}\eta^2 - \\ & G_{Vx}\frac{G_V\left(k(\theta - x_t)\right) + G_{Vx}\eta^2}{G_{VV}} \end{aligned} = 0$$

Dis-equilibrium

- Let $b = k(\theta - x_t)$. Rewrite:
- $G_t - G_V b \frac{G_V b + G_{Vx}\eta^2}{G_{VV}\eta^2} + G_x b + \frac{1}{2}\frac{\left[G_V b + G_{Vx}\eta^2\right]^2}{G_{VV}\eta^2} + \frac{1}{2}G_{xx}\eta^2 - G_{Vx}\frac{G_V b + G_{Vx}\eta^2}{G_{VV}} = 0$
- Multiply by $G_{VV}\eta^2$.
- $G_t G_{VV}\eta^2 - G_V b(G_V b + G_{Vx}\eta^2) + G_x b G_{VV}\eta^2 + \frac{1}{2}[G_V b + G_{Vx}\eta^2]^2 + \frac{1}{2}G_{xx}G_{VV}\eta^4 - G_{Vx}\eta^2[G_V b + G_{Vx}\eta^2] = 0$

Simplification

- Note that
- $-G_V b(G_V b + G_{Vx}\eta^2) + \frac{1}{2}[G_V b + G_{Vx}\eta^2]^2 - G_{Vx}\eta^2[G_V b + G_{Vx}\eta^2] = -\frac{1}{2}(G_V b + G_{Vx}\eta^2)^2$
- The PDE becomes
- $G_t G_{VV}\eta^2 + G_x b G_{VV}\eta^2 + \frac{1}{2}G_{xx}G_{VV}\eta^4 - \frac{1}{2}(G_V b + G_{Vx}\eta^2)^2 = 0$

The Partial Differential Equation (3)

- $G_t G_{VV} \eta^2 + G_x b G_{VV} \eta^2 + \frac{1}{2} G_{xx} G_{VV} \eta^4 - \frac{1}{2}(G_V b + G_{Vx} \eta^2)^2 = 0$

Ansatz for G

- $G(t, V, x) = f(t, x)V^{\gamma}$
- $G(T, V, x) = V^{\gamma}$
- $f(T, x) = 1$
- $G_t = V^{\gamma} f_t$
- $G_V = \gamma V^{\gamma-1} f$
- $G_{VV} = \gamma(\gamma - 1)V^{\gamma-2} f$
- $G_x = V^{\gamma} f_x$
- $G_{Vx} = \gamma V^{\gamma-1} f_x$
- $G_{xx} = V^{\gamma} f_{xx}$

Another PDE (1)

- $V^{\gamma} f_t \gamma(\gamma-1)V^{\gamma-2} f\eta^2 + V^{\gamma} f_x b\gamma(\gamma-1)V^{\gamma-2} f\eta^2 + \frac{1}{2} V^{\gamma} f_{xx} \gamma(\gamma-1)V^{\gamma-2} f\eta^4 - \frac{1}{2}(\gamma V^{\gamma-1} f b + \gamma V^{\gamma-1} f_x \eta^2)^2 = 0$
- Divide by $\gamma(\gamma-1)\eta^2 V^{2\gamma-2}$.
- $f_t f + f_x b f + \frac{1}{2} f_{xx} f\eta^2 - \frac{\gamma}{2(\gamma-1)}\left(f\frac{b}{\eta} + f_x\eta\right)^2 = 0$

Ansatz for f

- $f f_t + b f f_x + \frac{1}{2}\eta^2 f f_{xx} - \frac{\gamma}{2(\gamma-1)}\left(\frac{b}{\eta} f + \eta f_x\right)^2 = 0$
- $f(t,x) = g(t)\exp[x\beta(t) + x^2\alpha(t)] = g\exp(x\beta + x^2\alpha)$
- $f_t =$
  $g_t \exp(x\beta + x^2\alpha) + g\exp(x\beta + x^2\alpha)\,(x\beta_t + x^2\alpha_t)$
- $f_x = g\exp(x\beta + x^2\alpha)\,(\beta + 2x\alpha)$
- $f_{xx} =$
  $g\exp(x\beta + x^2\alpha)\,(\beta + 2x\alpha)^2 + g\exp(x\beta + x^2\alpha)\,(2\alpha)$
- $\frac{f_x}{f} = \beta + 2\alpha x$

Boundary Conditions

- $f(T,x) = g(T)\exp[x\beta(T) + x^2\alpha(T)] = 1$
- $g(T) = 1$
- $\alpha(T) = 0$
- $\beta(T) = 0$

Yet Another PDE (1)

- $g\exp(x\beta + x^2\alpha)\,[g_t \exp(x\beta + x^2\alpha) + g\exp(x\beta + x^2\alpha)\,(x\beta_t + x^2\alpha_t)] +$
  $bg\exp(x\beta + x^2\alpha)\,g\exp(x\beta + x^2\alpha)\,(\beta + 2x\alpha) +$
  $\frac{1}{2}\eta^2 g\exp(x\beta + x^2\alpha)\,[g\exp(x\beta + x^2\alpha)\,(\beta + 2x\alpha)^2 + g\exp(x\beta + x^2\alpha)\,(2\alpha)] -$
  $\frac{\gamma}{2(\gamma-1)}\left(\frac{b}{\eta} g\exp(x\beta + x^2\alpha) + \eta g\exp(x\beta + x^2\alpha)\,(\beta + 2x\alpha)\right)^2 = 0$
- Divide by g $\exp(x\beta + x^2\alpha)\exp(x\beta + x^2\alpha)$.
- $[g_t + g(x\beta_t + x^2\alpha_t)] + bg(\beta + 2x\alpha) + \frac{1}{2}\eta^2[g(\beta + 2x\alpha)^2 + g(2\alpha)] -$
  $\frac{\gamma}{2(\gamma-1)} g\left(\frac{b}{\eta} + \eta(\beta + 2x\alpha)\right)^2 = 0$
- $g_t + g(x\beta_t + x^2\alpha_t) + bg(\beta + 2x\alpha) + \frac{1}{2}\eta^2 g(\beta + 2x\alpha)^2 + \eta^2 g\alpha -$
  $\frac{\gamma}{2(\gamma-1)} g\left(\frac{b}{\eta} + \eta(\beta + 2x\alpha)\right)^2 = 0$

Yet Another PDE (2)

- $\lambda = \frac{\gamma}{2(\gamma-1)}$
- $g_t + g(x\beta_t + x^2\alpha_t) + bg(\beta + 2x\alpha) + \frac{1}{2}\eta^2 g(\beta + 2x\alpha)^2 + \eta^2 g\alpha - \lambda g\left(\frac{b}{\eta} + \eta(\beta + 2x\alpha)\right)^2 = 0$

Expansion in x

- $g_t + gx\beta_t + gx^2\alpha_t + bg\beta + 2x\alpha bg + \frac{1}{2}\eta^2 g(\beta^2 + 4x^2\alpha^2 + 4x\alpha\beta) + \eta^2 g\alpha - \lambda g\left(\frac{b^2}{\eta^2} + \eta^2\beta^2 + 4\eta^2 x^2\alpha^2 + 2b\beta + 4bx\alpha + 4\eta^2 x\alpha\beta\right) = 0$
- $g_t + gx\beta_t + gx^2\alpha_t + k(\theta - x)g\beta + 2x\alpha k(\theta - x)g + \frac{1}{2}\eta^2 g(\beta^2 + 4x^2\alpha^2 + 4x\alpha\beta) + \eta^2 g\alpha - \lambda g\left(\frac{k^2(\theta-x)^2}{\eta^2} + \eta^2\beta^2 + 4\eta^2 x^2\alpha^2 + 2k(\theta - x)\beta + 4k(\theta - x)x\alpha + 4\eta^2 x\alpha\beta\right) = 0$
- $g_t + gx\beta_t + gx^2\alpha_t + kg\beta\theta - kg\beta x + 2x\alpha kg\theta - 2\alpha kgx^2 + \frac{1}{2}\eta^2 g\beta^2 + 2\eta^2 gx^2\alpha^2 + 2\eta^2 gx\alpha\beta + \eta^2 g\alpha - \frac{\lambda g}{\eta^2}k^2\theta^2 + 2\frac{\lambda g}{\eta^2}k^2\theta x - \frac{\lambda g}{\eta^2}k^2 x^2 - \lambda g\eta^2\beta^2 - 4\lambda g\eta^2 x^2\alpha^2 - 2\lambda gk\beta\theta + 2\lambda gk\beta x - 4\lambda gk\theta x\alpha + 4\lambda gkx^2\alpha - 4\lambda g\eta^2 x\alpha\beta = 0$

Grouping in x

- $\left(g_t + kg\beta\theta + \frac{1}{2}\eta^2 g\beta^2 + \eta^2 g\alpha - \frac{\lambda g}{\eta^2}k^2\theta^2 - \lambda g\eta^2\beta^2 - 2\lambda gk\beta\theta\right) +$
  $\left(g\beta_t - kg\beta + 2\alpha kg\theta + 2\eta^2 g\alpha\beta + 2\frac{\lambda g}{\eta^2}k^2\theta + 2\lambda gk\beta - 4\lambda gk\theta\alpha - 4\lambda g\eta^2\alpha\beta\right)x +$
  $\left(g\alpha_t - 2\alpha kg + 2\eta^2 g\alpha^2 - \frac{\lambda g}{\eta^2}k^2 - 4\lambda g\eta^2\alpha^2 + 4\lambda gk\alpha\right)x^2 = 0$

The Three PDE's (1)

- $g\alpha_t - 2\alpha k g + 2\eta^2 g\alpha^2 - \frac{\lambda g}{\eta^2}k^2 - 4\lambda g\eta^2\alpha^2 + 4\lambda g k\alpha = 0$
- $g\beta_t - kg\beta + 2\alpha k g\theta + 2\eta^2 g\alpha\beta + 2\frac{\lambda g}{\eta^2}k^2\theta + 2\lambda g k\beta - 4\lambda g k\theta\alpha - 4\lambda g\eta^2\alpha\beta = 0$
- $g_t + kg\beta\theta + \frac{1}{2}\eta^2 g\beta^2 + \eta^2 g\alpha - \frac{\lambda g}{\eta^2}k^2\theta^2 - \lambda g\eta^2\beta^2 - 2\lambda g k\beta\theta = 0$

PDE in α

- $\alpha_t + (2\eta^2 - 4\lambda\eta^2)\alpha^2 + (4\lambda k - 2k)\alpha - \frac{\lambda}{\eta^2}k^2 = 0$
- $\alpha_t = \frac{\lambda}{\eta^2}k^2 + 2k(1 - 2\lambda)\alpha + 2\eta^2(2\lambda - 1)\alpha^2$

PDE in β, α

- $\beta_t - k\beta + 2\eta^2\alpha\beta + 2\lambda k\beta - 4\lambda\eta^2\alpha\beta - 4\lambda k\theta\alpha + 2\frac{\lambda}{\eta^2}k^2\theta + 2\alpha k\theta = 0$
- $\beta_t = (k - 2\eta^2\alpha - 2\lambda k + 4\lambda\eta^2\alpha)\beta + \left(4\lambda k\theta\alpha - 2\frac{\lambda}{\eta^2}k^2\theta - 2\alpha k\theta\right)$

PDE in β, α, g

- $g_t + kg\beta\theta + \frac{1}{2}\eta^2 g\beta^2 + \eta^2 g\alpha - \frac{\lambda g}{\eta^2}k^2\theta^2 - \lambda g\eta^2\beta^2 - 2\lambda gk\beta\theta = 0$
- $g_t = -kg\beta\theta - \frac{1}{2}\eta^2 g\beta^2 - \eta^2 g\alpha + \frac{\lambda g}{\eta^2}k^2\theta^2 + \lambda g\eta^2\beta^2 + 2\lambda gk\beta\theta$
- $g_t = g\left(-k\beta\theta - \frac{1}{2}\eta^2\beta^2 - \eta^2\alpha + \frac{\lambda}{\eta^2}k^2\theta^2 + \lambda\eta^2\beta^2 + 2\lambda k\beta\theta\right)$

Riccati Equation

- A Riccati equation is any ordinary differential equation that is quadratic in the unknown function.
- $\alpha_t = \frac{\lambda}{\eta^2} k^2 + 2k(1 - 2\lambda)\alpha + 2\eta^2(2\lambda - 1)\alpha^2$
- $\alpha_t = A_0 + A_1 \alpha + A_2 \alpha^2$

Solving a Riccati Equation by Integration

- Suppose a particular solution, α_1, can be found.
- $\alpha = \alpha_1 + \frac{1}{z}$ is the general solution, subject to some boundary condition.

Particular Solution

- Either α_1 or α_2 is a particular solution to the ODE. This can be verified by mere substitution.
- $\alpha_{1,2} = \frac{-A_1 \pm \sqrt{{A_1}^2 - 4A_2A_0}}{2A_2}$

z Substitution

- Suppose $\alpha = \alpha_1 + \frac{1}{z}$.
- $\dot{\left(\frac{1}{z}\right)} = A_0 + A_1\left(\alpha_1 + \frac{1}{z}\right) + A_2\left(\alpha_1 + \frac{1}{z}\right)^2$
- $= A_0 + \left(A_1\alpha_1 + A_1\frac{1}{z}\right) + \left(A_2{\alpha_1}^2 + A_2\frac{1}{z^2} + 2A_2\frac{\alpha_1}{z}\right)$
- $= A_0 + \left(A_1\alpha_1 + A_1\frac{1}{z}\right) + \left(A_2{\alpha_1}^2 + A_2\frac{1}{z^2} + 2A_2\frac{\alpha_1}{z}\right)$
- $= \left(A_0 + A_1\alpha_1 + A_2{\alpha_1}^2\right) + \left(\frac{A_1 + 2\alpha_1 A_2}{z}\right) + \frac{A_2}{z^2}$

goes to 0 by the definition of α_1

Solving z

- $\dot{\left(\frac{1}{z}\right)} = \left(\frac{A_1+2\alpha_1 A_2}{z}\right) + \frac{A_2}{z^2}$
- $-\frac{1}{z^2}\dot{z} = \left(\frac{A_1+2\alpha_1 A_2}{z}\right) + \frac{A_2}{z^2}$
- 1st order linear ODE
  - $\dot{z} + (A_1 + 2\alpha_1 A_2)z = -A_2$
- $z(t) = \frac{-A_2}{A_1+2\alpha_1 A_2} + C\exp(-(A_1 + 2\alpha_1 A_2)t)$

Solving for α

- $\alpha = \alpha_1 + \frac{1}{\frac{-A_2}{A_1+2\alpha_1 A_2}+c\exp(-(A_1+2\alpha_1 A_2)t)}$
- boundary condition:
  - $\alpha(T) = 0$
- $\alpha_1 + \frac{1}{\frac{-A_2}{A_1+2\alpha_1 A_2}+c\exp(-(A_1+2\alpha_1 A_2)T)} = 0$
- $C\exp(-(A_1 + 2\alpha_1 A_2)T) = -\frac{1}{\alpha_1} + \frac{A_2}{A_1+2\alpha_1 A_2}$
- $C = \exp\big((A_1 + 2\alpha_1 A_2)T\big)\left[\frac{A_2}{A_1+2\alpha_1 A_2} - \frac{1}{\alpha_1}\right]$

α Solution (1)

- $\alpha = \alpha_1 + \frac{1}{\frac{-A_2}{A_1+2\alpha_1 A_2}+c\exp(-(A_1+2\alpha_1 A_2)t)}$
- $=$
  $\alpha_1 +$
  $\frac{1}{\frac{-A_2}{A_1+2\alpha_1 A_2}+\exp\left((A_1+2\alpha_1 A_2)T\right)\left[\frac{A_2}{A_1+2\alpha_1 A_2}-\frac{1}{\alpha_1}\right]\exp(-(A_1+2\alpha_1 A_2)t)}$
- $= \alpha_1 + \frac{1}{\frac{-A_2}{A_1+2\alpha_1 A_2}+\exp\left((A_1+2\alpha_1 A_2)(T-t)\right)\left[\frac{A_2}{A_1+2\alpha_1 A_2}-\frac{1}{\alpha_1}\right]}$
- $= \alpha_1 + \frac{\alpha_1(A_1+2\alpha_1 A_2)}{-\alpha_1 A_2+\exp\left((A_1+2\alpha_1 A_2)(T-t)\right)(\alpha_1 A_2-A_1-2\alpha_1 A_2)}$

α Solution (2)

- $\alpha = \alpha_1 + \frac{\alpha_1(A_1+2\alpha_1 A_2)}{-\alpha_1 A_2+\exp\left((A_1+2\alpha_1 A_2)(T-t)\right)(-A_1-\alpha_1 A_2)}$
- $= \alpha_1\left[1 - \frac{A_1+2\alpha_1 A_2}{A_2+\exp\left((A_1+2\alpha_1 A_2)(T-t)\right)\left(\frac{A_1}{\alpha_1}+A_2\right)}\right]$
- $= \alpha_1\left[1 - \frac{\frac{A_1}{A_2}+2\alpha_1}{1+\left(\frac{A_1}{\alpha_1 A_2}+1\right)\exp\left((A_1+2\alpha_1 A_2)(T-t)\right)}\right]$

α Solution (3)

- $\alpha(t) = \frac{k}{2\eta^2}\left[\left(1-\sqrt{1-\gamma}\right) + \frac{2\sqrt{1-\gamma}}{1+\left(1-\frac{2}{1-\sqrt{1-\gamma}}\right)\exp\left(\frac{2k}{\sqrt{1-\gamma}}(T-t)\right)}\right]$

Solving β

- $\beta_t = (k - 2\eta^2\alpha - 2\lambda k + 4\lambda\eta^2\alpha)\beta + \left(4\lambda k\theta\alpha - 2\frac{\lambda}{\eta^2}k^2\theta - 2\alpha k\theta\right)$
- Let $\tau = T - t$
- $\hat{\beta}(\tau) = \beta(T - t)$
- $\hat{\beta}_\tau(\tau) = -\beta_t(\tau)$
- $-\hat{\beta}_\tau(\tau) = \beta_t(\tau) = \left(k - 2\eta^2\alpha(\tau) - 2\lambda k + 4\lambda\eta^2\alpha(\tau)\right)\beta(\tau) + \left(4\lambda k\theta\alpha(\tau) - 2\frac{\lambda}{\eta^2}k^2\theta - 2\alpha(\tau)k\theta\right)$
- $\hat{\beta}_\tau(\tau) = (-k + 2\eta^2\hat{\alpha} + 2\lambda k - 4\lambda\eta^2\hat{\alpha})\hat{\beta} + \left(-4\lambda k\theta\hat{\alpha} + 2\frac{\lambda}{\eta^2}k^2\theta + 2\hat{\alpha}k\theta\right)$
- $\hat{\beta}_\tau(\tau) = \left((2\lambda - 1)k + 2\eta^2\hat{\alpha}(1 - 2\lambda)\right)\hat{\beta} + \left(2\hat{\alpha}k\theta(1 - 2\lambda) + 2\frac{\lambda}{\eta^2}k^2\theta\right)$

First Order Non-Constant Coefficients

- $\hat{\beta}_\tau = B_1\hat{\beta} + B_2$
  - $B_1(\tau) = (2\lambda - 1)k + 2\eta^2\hat{\alpha}(1 - 2\lambda)$
  - $B_2(\tau) = 2\hat{\alpha}k\theta(1 - 2\lambda) + 2\frac{\lambda}{\eta^2}k^2\theta$

Integrating Factor (1)

- $\hat{\beta}_\tau - B_1\hat{\beta} = B_2$
- We try to find an integrating factor $\mu = \mu(\tau)$ s.t.
  - $\frac{d}{d\tau}\left(\mu\hat{\beta}\right) = \mu\frac{d\widehat{\beta}}{d\tau} + \hat{\beta}\frac{d\mu}{d\tau} = \mu B_2$
- Divide LHS and RHS by $\mu\hat{\beta}$.
  - $\frac{1}{\widehat{\beta}}\frac{d\widehat{\beta}}{d\tau} + \frac{1}{\mu}\frac{d\mu}{d\tau} = \frac{B_2}{\widehat{\beta}}$
- By comparison,
  - $-B_1 = \frac{1}{\mu}\frac{d\mu}{d\tau}$

Integrating Factor (2)

- $\int -B_1 d\tau = \int \frac{d\mu}{\mu} = \log \mu + C$
- $\mu = \exp\left(\int -B_1 d\tau\right)$
- $\mu\hat{\beta} = \int \mu B_2 d\tau + C$
- $\hat{\beta} = \frac{\int \mu B_2 d\tau + C}{\mu}$
- $\hat{\beta} = \frac{\int \exp\left(\int -B_1 du\right) B_2 d\tau + C}{\exp\left(\int -B_1 d\tau\right)}$

$\hat{\beta}$ Solution

- $\hat{\beta} = \frac{\int_0^\tau \exp\left(\int_0^\tau -B_1(u)du\right)B_2(s)ds}{\exp\left(\int_0^\tau -B_1(u)du\right)} + C$
- $\hat{\beta}(\tau) =$
  $\exp\left(\int_0^\tau B_1(u)du\right) \int_0^\tau \left[\exp\left(\int_0^s -B_1(u)du\right) B_2(s)\right]ds + C$

B_1, B_2

- $\int_0^\tau B_1(s)ds = \int_0^\tau [(2\lambda - 1)k + 2\eta^2(1 - 2\lambda)\alpha(s)]ds$
- $I_2 = \int_0^\tau \left[\exp\left(\int_0^s -B_1(u)du\right) B_2(s)\right]ds$

β Solution

- $$\beta(t) = \frac{k\theta}{\eta^2}\left(1 + \sqrt{1-\gamma}\right)\frac{\exp\left(\frac{2k}{\sqrt{1-\gamma}}(T-t)\right)-1}{1+\left[1-\frac{2}{1-\sqrt{1-\gamma}}\right]\exp\left(\frac{2k}{\sqrt{1-\gamma}}(T-t)\right)}$$

Solving g

- $g_t = g\left(-k\beta\theta - \frac{1}{2}\eta^2\beta^2 - \eta^2\alpha + \frac{\lambda}{\eta^2}k^2\theta^2 + \lambda\eta^2\beta^2 + 2\lambda k\beta\theta\right)$
- Withα and β solved, we are now ready to solve g_t.
- $\frac{g_t}{g} = G(t)$
- $\frac{d}{ds}(\log g_t) = \frac{g_t}{g} = G$
- $\log g_t = \int G ds + C$
- $g_t = C\exp(\int G dt) = \exp\left(-\int_0^T G ds\right)\exp\left(\int_0^t G ds\right)$
- $g_t = \exp\left(-\int_t^T G ds\right)$

Computing the Optimal Position

- $h(t)^* = -\frac{G_V(k(\theta - x_t)) + G_{Vx}\eta^2}{G_{VV}\eta^2}$
- $= -\frac{\gamma V^{\gamma-1} f(k(\theta - x_t)) + \gamma V^{\gamma-1} f_x \eta^2}{\gamma(\gamma-1)V^{\gamma-2} f \eta^2}$
- $= -\frac{V f(k(\theta - x_t)) + V f_x \eta^2}{(\gamma-1) f \eta^2}$
- $= -\frac{V}{(\gamma-1)\eta^2} \frac{f(k(\theta - x_t)) + f_x \eta^2}{f}$
- $= -\frac{V}{(\gamma-1)\eta^2} \left[k(\theta - x_t) + \frac{f_x}{f}\eta^2\right]$
- $= \frac{V}{(1-\gamma)\eta^2} [k(\theta - x_t) + \eta^2(\beta + 2\alpha x)]$
- $= \frac{V}{(1-\gamma)} \left[-\frac{k}{\eta^2}(x_t - \theta) + 2\alpha x + \beta\right]$

The Optimal Position

- $h(t)^* = \frac{V_t}{(1-\gamma)}\left[-\frac{k}{\eta^2}(x_t - \theta) + 2\alpha(t)x_t + \beta(t)\right]$
- $h(t)^* \sim -\frac{k}{\eta^2}(x_t - \theta)$

P&L for Simulated Data

- The portfolio increases from $1000 to $4625 in one year.

Parameter Estimation

- Can be done using Maximum Likelihood.
- Evaluation of parameter sensitivity can be done by Monte Carlo simulation.
- In real trading, it is better to be conservative about the parameters.
  - Better underestimate the mean-reverting speed
  - Better overestimate the noise
- To account for parameter regime changes, we can use:
  - a hidden Markov chain model
  - moving calibration window